Heat kernel estimates and parabolic Harnack inequalities for symmetric Dirichlet forms

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This is based on a joint work with Zhen-Qing Chen and Takashi Kumagai

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Outline

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Symmetric Dirichlet form (diffusion with jumps)

- Let (M, d, μ) be a *metric measure space*.
- Consider a regular *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on $L^2(M;\mu)$ as follows:

$$\begin{aligned}\mathcal{E}(f,g) =& \mathcal{E}^{(c)}(f,g) + \iint_{M\times M} (f(x)-f(y))(g(x)-g(y))\,J(dx,dy)\\ =: & \mathcal{E}^{(c)}(f,g) + \mathcal{E}^{(j)}(f,g),\end{aligned}$$

  where $(\mathcal{E}^{(c)}, \mathcal{F})$ is the strongly local part of $(\mathcal{E}, \mathcal{F})$, and $J(\cdot,\cdot)$ is a symmetric Radon measure $M \times M \setminus$ diag.

- The associated integro-differential operator $\mathcal{L}$ which satisfies that

$$\langle -\mathcal{L}f, g\rangle = \mathcal{E}(f,g).$$

- Transition density function

$$\mathbb{E}^x f(X_t) = P_t f(x) = \int p(t,x,y) f(y)\,\mu(dy), \quad x \in M_0, f \in L^\infty(M;\mu).$$

- Fundamental solution

$$\frac{\partial}{\partial t} p(t,x,y) = \mathcal{L}p(t,\cdot,y)(x).$$

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$$\Delta + \Delta^{\alpha/2}.$$

- Chen-Kugamai ('10): General diffusions with jumps:

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^d} \nabla f(x)\cdot A(x)\nabla f(x)\,dx + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x)-f(y)^2)}{|x-y|^{d+\alpha}} c(x,y)\,dx\,dy.$$

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Question: inner uniform domains in Harnack-type Dirichlet spaces

Example (Gyrya-Saloff-Coste, '11)

Let $U \subset \mathbb{R}^d$ be an unbounded global Lipschitz domain equipped with the Euclidean distance. Let $X = \{X_t\}_{t\geqslant 0}$ be a symmetric reflected diffusion with jumps on $\overline{U}$ associated with the regular Dirichlet form $(\mathcal{E}, W^{1,2}(U))$ on $L^2(U; dx)$ given by

$$\mathcal{E}(u,v) = \int_U \nabla u(x) \cdot A(x)\nabla v(x)\, dx + \int_U \int_U (u(x) - u(y))(v(x) - v(y)) \frac{c(x,y)}{|x-y|^{d+\alpha}}\, dx\, dy,$$

where $A(x) = (a_{ij}(x))_{1\leqslant i,j\leqslant d}$ is a measurable uniformly elliptic and bounded $d \times d$ matrix-valued function on U, $0 < \alpha < 2$, and $c(\cdot,\cdot)$ is a symmetric measurable function on $U \times U$ that is bounded between two positive constants. Its L^2-infinitesimal generator is of the form

$$\mathcal{L}u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u(x)}{\partial x_j}\right) + 2 \lim_{\varepsilon\to 0} \int_{\{y\in U:\, |y-x|>\varepsilon\}} (u(y) - u(x)) \frac{c(x,y)}{|x-y|^{d+\alpha}}\, dy$$

with "Neumann" boundary condition.

Example

Suppose that (M, d, μ) is an Alfhors d-regular set. Consider a diffusion with jumps whose corresponding Dirichlet form is given by

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_M \int_M \frac{(u(x)-u(y))(v(x)-v(y))}{d(x,y)^{d+\alpha}} c(x,y)\, \mu(dx)\, \mu(dy),$$

where $\mathcal{E}^{(c)}(\cdot,\cdot)$ is a strongly local regular Dirichlet form which enojys sub-Gaussian estimates with the walk dimension $\beta > 2$, (for example, a Brownian motion on the D-dimensional unbounded Sierpinski gasket, see Barlow-Perkins ('88)), and $0 < \alpha < \beta$.

Outline

1. Introduction
   - Setting
   - Purpose
2. Heat kernel estimates for local / nolocal Dirichlet forms
   - Local Dirichlet forms
   - Non-local Dirichlet forms
3. Main results
   - Heat kernel estimates
   - Parabolic Harnack inequalities

Purpose

- Stable characterizations of both upper bounds and two-sided estimates on heat kernel for symmetric Dirichlet forms including both local and non-local terms on general metric measure spaces.
- Stable characterizations of parabolic Harnack inequalities.
- One of the difficulties in obtaining fine properties for diffusions with jumps and associated operators is that it exhibits different scales: the strongly local terms part has a diffusion scaling $r \mapsto \phi_c(r)$ while the pure jump part has a different type of scaling $r \mapsto \phi_j(r)$. (For example, $\phi_c(r) = r^2$ for Δ, and $\phi_j(r) = r^\alpha$ for $\Delta^{\alpha/2}$.)
- With $\phi(r) := \phi_c(r) \wedge \phi_j(r)$,

$$HK_-(\phi_c, \phi_j) \Longleftrightarrow PHI(\phi) + J_{\phi_j}.$$

This assertion is different from diffusion cases, where it holds that

$$HK(\phi_c) \Longleftrightarrow PHI(\phi_c);$$

it is also different from pure jump cases, where it holds that

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Outline

Gaussian estimates

- Let

$$\mathcal{L} := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

be a uniformly elliptic div. form on $\mathbb{R}^d$ so that

$$\sigma^{-1} I \leqslant (a_{ij}(x))_{1\leqslant i,j\leqslant d} \leqslant \sigma I, \quad x \in \mathbb{R}^d$$

with some constant $\sigma \geqslant 1$.

- $$\frac{c_1}{t^{d/2}} \exp\left(-c_2\frac{|x-y|^2}{t}\right) \leqslant p(t,x,y) \leqslant \frac{c_3}{t^{d/2}} \exp\left(-c_4\frac{|x-y|^2}{t}\right),$$

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- Heat kernel estimates (HK(2)):
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- Local regular DF
  $$HK(2) \Leftrightarrow VD + PI(2) \Leftrightarrow PHI(2),$$
  see Grigor'yan ('91), Saloff-Coste ('92), Sturm ('96).
  - VD: $V(x,2r) \leqslant C_\mu V(x,r)$, where $V(x,r) = \mu(B(x,r))$.
  - $PI(2)$:
    $$\int_{B_r} (f - \bar{f}_{B_r})^2 \, d\mu \leqslant Cr^2 \int_{B_{\kappa r}} d\Gamma_c(f,f).$$

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Sub-Gaussian estimates

- Diffusions on 'nice' fractals M: $\exists d_w \geqslant 2$ s.t. HK(d_w) holds:

$$\frac{c_1}{\mu(B(x,t^{1/d_w}))}\exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$
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- Barlow-Bass, Barlow-Bass-Kumagai, Andres-Barlow, Grigor'yan-Hu-Lau:

$$HK(d_w)\Leftrightarrow VD+PI(d_w)+CS(d_w)\Leftrightarrow PHI(d_w).$$

  - $CS(d_w)$: for any $x_0\in M$, $0<r\leqslant R$ and $f\in\mathcal{F}$, there is a cut-off function ψ on $B(x_0,R)\subset B(x_0,R+r)$,

$$\int_{B(x_0,R+r)} f^2\,d\Gamma_c(\psi,\psi)\leqslant C_1\int_{B(x_0,R+r)}\psi^2\,d\Gamma_c(f,f)+\frac{C_2}{r^{d_w}}\int_{B(x_0,R+r)} f^2\,d\mu.$$

  - The De Giorgi-Nash-Moser theory in PDE.

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More general heat kernel estimates

- Let $\phi_c : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous function such that there exist constants $c_{1,\phi_c}, c_{2,\phi_c} > 0$ and $\beta_{2,\phi_c} \geqslant \beta_{1,\phi_c} > 1$ such that

$$c_{1,\phi_c}\left(\frac{R}{r}\right)^{\beta_{1,\phi_c}} \leqslant \frac{\phi_c(R)}{\phi_c(r)} \leqslant c_{2,\phi_c}\left(\frac{R}{r}\right)^{\beta_{2,\phi_c}} \quad \text{for all } 0 < r \leqslant R.$$

- Andres-Barlow ('15): Davies-Gaffney's estimate

$$p^{(c)}(t,x,y) = \frac{1}{V(x,\phi_c^{-1}(t))} \exp\left(-\sup_{s>0}\left\{\frac{d(x,y)}{s} - \frac{t}{\phi_c(s)}\right\}\right).$$

- Hambly-Kumagai ('99); Grigor'yan-Telcs ('12): Chaining argument

$$p^{(c)}(t,x,y) = \frac{1}{V(x,\phi_c^{-1}(t))} \exp\left(-\frac{d(x,y)}{\bar{\phi}_c^{-1}(t/d(x,y))}\right),$$

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Jump processes: $0 < \alpha < 2$

- Symmetric α-stable-like process (Chen-Kumagai ('03)) – This corresponds to div. form.

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on a d-set M, where

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- $$c_1\left(t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}}\right) \leqslant p(t,x,y) \leqslant c_2\left(t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}}\right)$$

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Many people have been working on related topics.

Barlow, Bae, Bañuelos, Bass, Bendikov, Bogdan, Caffarelli, Chen, Cygan, Dyda, Fukushima, Grigor'yan, Grzywny, Guan, Hasebe, Hoh, E. Hu, J. Hu, Husseini, Ishikawa, Jacob, Jakubowski, Kang, Kassman, Kigami, Kim, Knopova, Kolokoltsov, Komatsu, Kühn, Kulczycki, Kumagai, Kuwae, Kwaśnicki, Lau, Lee, Levin, Ma, Maejima, Maekawa, Masamune, Mimica, Miura, Murugan, Priola, Ren, Ryzner, Sakuma, Saloff-Coste, Sato, Schilling, Silvestre, Shiozawa, Song, Stós, Sztonyk, Takeda, Takeuchi, Uemura, Vondraček, Wada, W., L.Wang, Yang, Zähle, Zhang, $\cdots$

Jump processes: (Chen-Kumagai('08))

- J_{ϕ_j}:
$$J(x,y) \asymp \frac{1}{V(d(x,y))\phi_j(d(x,y))},$$
where $\mu(B(x,r)) \asymp V(r)$ such that *VD* and *RVD* are satisfied.
- $J_{\phi_j} \Leftrightarrow HK(\phi_j)$:
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for all $t > 0$ and $x, y \in M$.
- Example: $[\alpha_1, \alpha_2] \subset (0,2)$, ν is a probability measure on $[\alpha_1, \alpha_2]$ such that
$$\phi_j(t) := \left(\int_{\alpha_1}^{\alpha_2} t^{-\alpha}\,\nu(d\alpha)\right)^{-1}.$$
Especially, $\mu(B(x,r)) \asymp r^d$, $0 < \alpha_1 < \cdots < \alpha_n < 2$,
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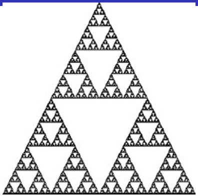
Jump processes: $\alpha > 2$.



Question

Let M be a Sierpinski gasket on $\mathbb{R}^2$, and $\mu(B(x,r)) \asymp r^d$ with $d = \frac{\log 3}{\log 2}$. Let

$$\mathcal{E}(f) \asymp \iint_{M\times M} \frac{(f(x)-f(y))^2}{d(x,y)^{d+\alpha}}\,\mu(dx)\,\mu(dy),$$

where $\alpha \in (0, \frac{\log 5}{\log 2})$ (possibly $\alpha \geqslant 2$). What is the expression of HK?

$$p(t,x,y) \asymp \frac{1}{t^{d/\alpha}} \wedge \frac{t}{d(x,y)^{d+\alpha}} ???$$

Jump processes

- $$\mathcal{E}(f,g)=\iint_{M\times M}(f(x)-f(y))(g(x)-g(y))\,J(dx,dy).$$

- J_{ϕ_j}:
$$J(x,y)\asymp\frac{1}{V(x,d(x,y))\phi_j(d(x,y))}.$$

- MMS (M,d,μ). Let $V(x,r)=\mu(B(x,r))$ for all $x\in M$ and $r>0$.

- VD and RVD
$$c_1\left(\frac{R}{r}\right)^{d_1}\leqslant\frac{V(x,R)}{V(x,r)}\leqslant c_2\left(\frac{R}{r}\right)^{d_2},\quad x\in M,0<r<R.$$

- Scaling function ϕ_j:
$$c_3\left(\frac{R}{r}\right)^{\beta_{1,j}}\leqslant\frac{\phi_j(R)}{\phi_j(r)}\leqslant c_4\left(\frac{R}{r}\right)^{\beta_{2,j}},\quad 0<r<R.$$

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- Example: Symmetric α-stable processes on $\mathbb{R}^d$,

$$J(x,y) \asymp \frac{1}{|x-y|^{d+\alpha}}$$

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Main result: heat kernel estimates

Theorem (Chen-Kumagai-W., '18)

The following are equivalent:

(i) $HK(\phi_j)$.

(ii) J_{ϕ_j} and $CS(\phi_j)$.

- $CS(\phi_j)$: For $0 < r \leqslant R, f \in \mathcal{F}$ and almost all $x \in M$, there exists a cutoff function φ for $B(x,R) \subset B(x,R+r)$ so that the following holds:

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Outline

Setting

- Consider a regular *Dirichlet form* $(\mathcal{E},\mathcal{F})$ on $L^2(M;\mu)$ as follows:

$$\begin{aligned} D(f,g) =& \mathcal{E}^{(c)}(f,g) + \iint_{M\times M} (f(x)-f(y))(g(x)-g(y))\,J(dx,dy) \\ =:& \, \mathcal{E}^{(c)}(f,g) + \mathcal{E}^{(j)}(f,g), \end{aligned}$$

  where $(\mathcal{E}^{(c)},\mathcal{F})$ is the strongly local part of $(\mathcal{E},\mathcal{F})$, and $J(\cdot,\cdot)$ is a symmetric Radon measure $M\times M\setminus$ diag.

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$$p^{(c)}(t,x,y) = \frac{1}{V(x, \phi_c^{-1}(t))} \exp\left(-\frac{d(x,y)}{\bar{\phi}_c^{-1}(t/d(x,y))}\right),$$

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- $HK_-(\phi_c, \phi_j)$: the lower bound is replaced by

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Heat kernel estimates

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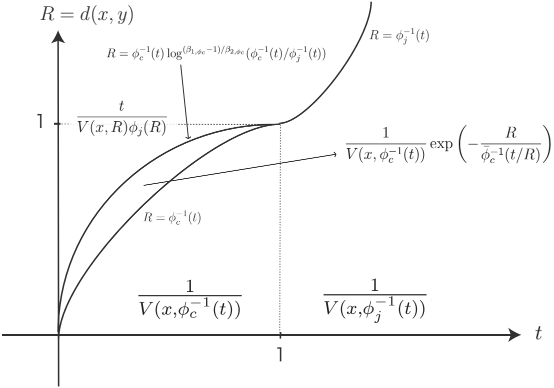

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Main result: heat kernel estimates

Theorem (Chen-Kumagai-W.,'19+)

Assume that the metric measure space (M, d, μ) satisfies VD and RVD, and that ϕ_c and ϕ_j satisfy the weak scaling property. Then the following are equivalent:

(1) $HK_-(\phi_c, \phi_j)$.

(2) $PI(\phi)$, J_{ϕ_j} and $Gcap(\phi)$.

(3) $PI(\phi)$, J_{ϕ_j} and $CS(\phi)$,

where

$$\phi(r) := \phi_c(r) \wedge \phi_j(r) = \begin{cases} \phi_c(r), & r \in (0, 1], \\ \phi_j(r), & r \in [1, \infty). \end{cases}$$

If, additionally, (M, d, μ) is connected and satisfies the chain condition, then all the conditions above are equivalent to:

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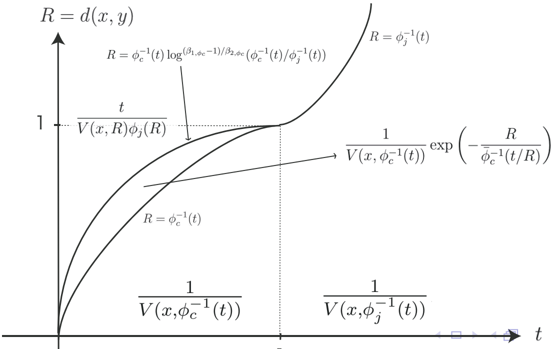
Proofs

- Stability results via $CS(\phi)$ from (Chen-Kuamgai-W., '18).
- The most difficult part is $UHKD(\phi) + J_{\phi_j,\leqslant} + E_\phi \Rightarrow UHK(\phi_c, \phi_j)$.
- We find a new self-improving argument with truncation apparoch for upper bounds for diffusions with jumps. The advantage of this technique is that it not only can take care of different scales both from the strongly local term part and the pure jump term, but also can treat the case that the volume of balls is not uniformly comparable.



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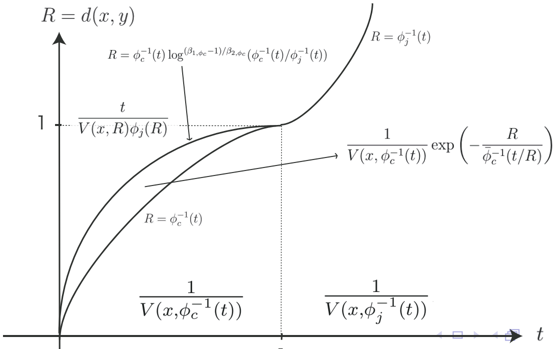

Outline

Main result: parabolic Harnack inequalities

Theorem (Chen-Kumagai-W.,'19+)

Suppose that the metric measure space (M, d, μ) satisfies VD and RVD, and that ϕ_c and ϕ_j satisfy the weak scaling property. The following statements are equivalent.

(1) $PHI(\phi)$.

(2) $PI(\phi) + J_{\phi,\leqslant} + Gcap(\phi) + UJS$.

(3) $PI(\phi) + J_{\phi,\leqslant} + CS(\phi) + UJS$.

In particular,

$$HK_-(\phi_c, \phi_j) \Longleftrightarrow PHI(\phi) + J_{\phi_j}.$$

- *UJS*: For almost all $x, y \in M$,

$$J(x,y) \leqslant \frac{c}{V(x,r)} \int_{B(x,r)} J(z,y)\,\mu(dz), \quad r \leqslant \frac{1}{2} d(x,y),$$

see Barlow-Bass-Kumagai ('09).

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Counterexample

Example ($PHI(\phi)$ alone does not imply $J_{\phi_j,\leqslant}$)

Let $M = \mathbb{R}^d$, and

$$J(x,y) \asymp \begin{cases} \frac{1}{|x-y|^{d+\alpha}} & |x-y| \leqslant 1; \\ \frac{1}{|x-y|^{d+\beta}} & |x-y| \geqslant 1, \end{cases}$$

where $\alpha, \beta \in (0,2)$. We consider the following regular Dirichlet form

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \nabla f(x) \cdot A(x) \nabla g(x)\, dx + \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) J(x,y)\, dx\, dy$$

and $\mathcal{F} = \overline{C_1(\mathbb{R}^d)}^{\mathcal{E}_1}$, where $A(x)$ is a measurable $d \times d$ matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic and bounded. Then, $PHI(\phi)$ holds with $\phi(r) = r^\beta \wedge r^2$ for all $r > 0$. (Note that ϕ does not depend on the choice of $\alpha \in (0,2)$.) Since $PHI(\phi)$ holds regardless of the choice of $\alpha \in (0,2)$, $PHI(\phi)$ alone does not imply the upper bound of the jumping kernel.

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Thank you for your attention!

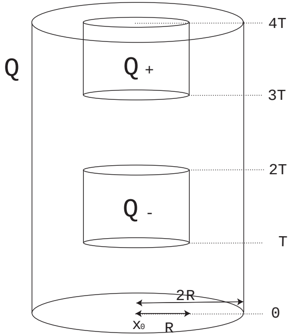
PHI(d_w)

Let $Q := (0, 4T) \times B(x_0, 2R)$. For $Q \subset M$, $u(t,x) : M \to \mathbb{R}_+$ is caloric on Q, if

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x), \qquad \forall t \in Q.$$

We say *PHI(d_w)* (**parabolic Harnack inequality**) holds, if $\exists C_1 > 0$ s.t. $\forall u = u(t,x)$ caloric and $\geqslant 0$ in Q with $T = R^{d_w}$, then

$$\sup_{Q_-} u \leqslant C_1 \inf_{Q_+} u.$$



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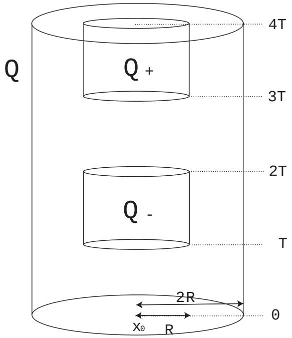

Proposition

Assume PHI(d_w)*. Then,* $\forall u$ *bounded and caloric in* $Q(x_0, r^{d_w}, r)$*, the following inequality is satisfied*

$$|u(t',x') - u(t'',x'')| \leqslant C \left(\frac{|t'-t''|^{1/d_w} + d(x',x'')}{r} \right)^{\gamma} \sup_{Q(x_0, r^{d_w}, r)} u$$

for $dt \times \mu$*-a.e.* $(s',x'), (s'',x'') \in Q(x_0, \delta r^{d_w}, \delta r)$*.*

- The De Giorgi-Nash-Moser theory in PDE.

Proposition

Assume PHI(d_w). Then, $\forall u$ bounded and caloric in $Q(x_0, r^{d_w}, r)$, the following inequality is satisfied

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- The De Giorgi-Nash-Moser theory in PDE.

Counterexample: $HK(\phi_j)$

Example (J_{ϕ_j} only does not imply $HK(\phi_j)$.)

Let $M = \mathbb{R}^d$, $\phi_j(r) = r^\alpha + r^\beta$ with $0 < \alpha < 2 < \beta$, and

$$J(x,y) \asymp \frac{1}{|x-y|^d \phi_j(|x-y|)}, \quad x, y \in \mathbb{R}^d.$$

Then, J_{ϕ_j} holds, but $HK(\phi_j)$ does not hold.

- $CS(\phi_j^*)$ holds with $\phi_j^*(r) \asymp r^\alpha + r^2$.
- $$HK(\phi_j) \Longleftrightarrow J_{\phi_j} \ + \ CS(\phi_j).$$

Main result: parabolic Harnack inequalities

Theorem (Chen-Kumagai-W., '18)

The following are equivalent:

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(ii) $J_{\phi_j,\leqslant}$, *UJS*, $CSJ(\phi_j)$ *and* $PI(\phi_j)$.

- $J_{\phi_j,\leqslant}$:

$$J(x,y) \leqslant \frac{c}{V(x,d(x,y))\phi_j(d(x,y))}.$$

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see Barlow-Bass-Kumagai ('09).

Corollary

$$HK(\phi_j) \Longleftrightarrow PHI(\phi_j) + J_{\phi_j,\geqslant}.$$

Counterexample

Example (Dyda-Kassmann, '15; $PHI(\phi_j)$ only does not imply $HK(\phi_j)$.)

Let $M = \mathbb{R}^d$ and $0 < \alpha < 2$. For $0 < \theta < 1$ and $v \in \mathbb{R}^d$ with $|v| = 1$, define $A = \{h \in \mathbb{R}^d : |(h/|h|, v)| \geqslant \theta\}$ and

$$J(x, y) = 1_A(x - y)|x - y|^{-d-\alpha}.$$

Then, $PHI(\phi_j)$ holds, but $HK(\phi_j)$ does not hold.

Hint: Both UJS and $PI(\phi_j)$ hold!

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