Helly graphs and Helly groups

9/5/2019 Lecture 2 :

Continuation of Isbell's proof :

(8) **Lemma 3** : If $e(X) \subseteq Y \subseteq E(X)$, and Y is injective, then $Y = E(X)$.

Proof: $\alpha : E(X) \to E(X)$ 1-Lip. s.t. α is ~~idempotent~~ identity on $e(X)$.

We claim that α is trivial.

Let $f \in E(X)$ and $g := \alpha(f)$.

$$g(x) \overset{(\text{by}(c))}{=} d_\infty(g, e(x)) \underset{\text{1-Lip}}{\leq} d_\infty(f, e(x)) \underset{(\text{by }(c))}{=} f(x). \Rightarrow g = f. \ \square$$

$\therefore \nexists$ Injective $Y = E(X)$

Suppose $|X| = n$. Then $e : X \to \mathbb{R}^n$.

$\Delta(X) = \{ f : f(x) + f(y) \geq d(x,y) \}$ is a convex unbounded, polyhedron.

$E(X) \to$ union of compact faces of $\Delta(X)$.

i.e., $E(X) = \bigcap_{\substack{e(x) \\ x \in X}} \left(\bigcup_{\substack{e(y) \\ y \in X}} I(e(x), e(y)) \right)$ $\to$ octahedron

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3. Helly graphs

Thm A: A graph H is Helly iff for any graph G, the pair (G, H) has the extension property.

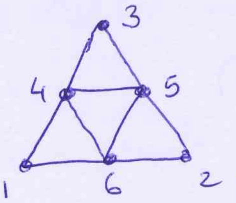
Graphs are always simplicial, and viewed as metric spaces on the vertices.

Thm B: For any graph G, $\exists$ the smallest Helly graph $\text{Helly}(G)$ into which G isometrically embeds.

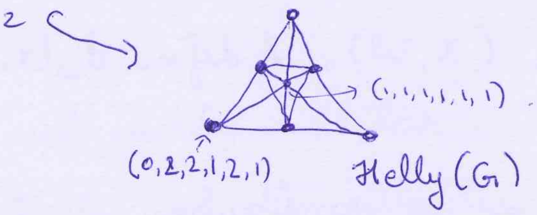
Helly (G) - Hellification of G.

$$V(\mathrm{Helly}(G)) = \{ f \in E(G, d_G) \text{ integer valued} \}$$

$$E(\mathrm{Helly}(G)) = \{ fg : d_\infty(f, g) = 1 \}$$

Eg: $G =$ 

Kuratowski image of $1 : (0, 2; 2, 1, 2, 1)$.



<u>Cubulation</u>

A wall-space is a set X with a collection of walls

$$\mathcal{W} = \{ (A_i, A_i') : A_i \cap A_i' = \emptyset,\ A_i \cup A_i' = X \}$$ and such that

$\forall\, x, y \in X$, $\exists$ finitely many walls separating them.

For any wall-space, one associates canonically a CAT(0) cube complex.

Half-spaces of CAT(0) cube complexes are components of hyperplane complements.

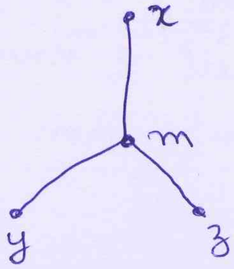
Half-spaces are gated sets in the 1-skeleton (with graph metric).

<u>Median graph</u>: Modular graphs with uniqueness of medians

i.e. $m : V^3 \to V$

$(x, y, z) \mapsto$ median of $\{x, y, z\}$.

median : point in $[x, y] \cap [y, z] \cap [z, x]$



$(V, m) \to$ median algebra

A median algebra is a set along with a ternary operation m

s.t. 1) $m(u, v, v) = v$.

2) $m(u, v, w) = m(u, w, v) = m(v, u, w)$

3) $m(u, v, m(u, w, x)) = m(u, m(u, v, w), x)$.

Given a wall-space $(X, \mathcal{W})$, define $d_w(x, y) = \#$ walls separating x & y.

$$(X, d_w) \xhookrightarrow{\alpha} \text{Boolean cube}.$$

Fix $x_0 \in X$. $\alpha(x) =$ all walls separating x_0 from x.

$$d_w(x, y) = |\alpha(x) \,\Delta\, \alpha(y)|$$

Given a graph G, $X_{clique}(G)$, the clique complex of G is obtained by filling each clique with a simplex (whose 1-skeleton is the clique).

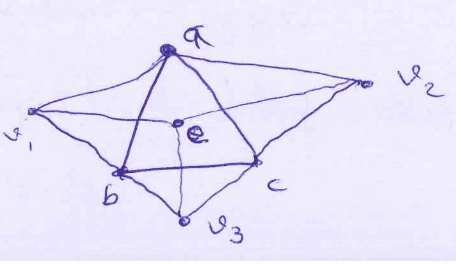
$X_\Delta(G) = X^{(2)}_{clique}(G)$.

$X_{cube}(G) =$ fill in cubes for each 1-skeleton of cube in G.

$X_\square(G) - X^{(2)}_{cube}(G)$

Helly graphs and clique-Helly graphs.

(Berge-Duchet)
Lemma 1: G is clique-Helly iff $\forall$ triangle $T = \{a, b, c\}$, the vertices v having at least 2 neighbours in T have at least one common neighbour e.



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<u>Thm 1</u> (CCHO) Let G be (finitely) clique-Helly and let $\widetilde{G}$ be the 1-skeleton of the universal cover $\widetilde{X}$ of $X = X_\Delta(G)$. Then $\widetilde{G}$ is Helly. Consequently, a graph is Helly iff $X_\Delta(G)$ is simply-connected and G clique-Helly.

<u>Thm 2</u>: For a graph G, TFAE.

(i) G is Helly.

(ii) G is 1-Helly and weakly-modular.

(iii) G is clique-Helly and dismantlable. (G loc. finite).

(iv) G is clique-Helly and $X_\Delta(G)$ is simply connected.

(v) G is clique-Helly ~~and~~ with contractible $X_{clique}(G)$ in the case of finite dimensional $X_{clique}(G)$.

1-Helly: Helly property for 1-balls.

<u>Properties of Helly graphs</u>

1. Helly $\Rightarrow$ clique-Helly.

Let C be maximal clique. Then $C = \bigcap_{w \in C} B_1(w)$.

Since Helly property holds for families produced by intersection, we are done.

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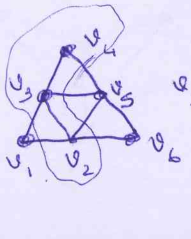
2. Helly $\Rightarrow$ dismantlability. (assume locally finite G).

A <u>dismantling order</u> on G is a total order $<$ on vertices.

$v_1, v_2, \ldots$ s.t. $\forall v_i$, $\exists v_j \sim v_i$, $j < i$ so that if ~~$v_k < v_i$, $v_k \sim v_i$~~

$v_k < v_i$, $v_k \sim v_i \Rightarrow$ ~~$v_k < v_j$ or $k = j$~~ $v_k \sim v_j$. ($\sim$: adjacent).

v_j dominates v_i



v_3 dominates v_5

Lemma 2: If G is dismantlable, then $X_{clique}(G)$ is contractible ($\therefore$ also simply-connected).

Lemma 3: If G is Helly, then any base-point order is a dismantling.

Let $v \in G$ be a vertex. A base-point ordering on G at v is any labelling of the vertices of G such that for any vertices u, w,

$$d(u,v) < d(w,v) \Rightarrow u < w.$$

There is no condition on the order of u, w if $d(u,v) = d(w,v)$.

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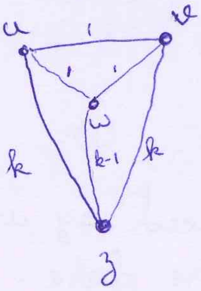
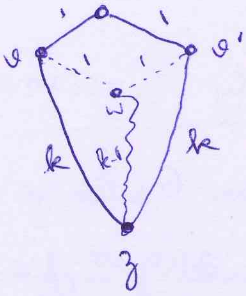
3. Helly $\Rightarrow$ weak (pseudo) - modularity.

A graph is weakly-modular if it satisfies

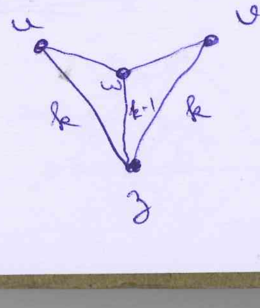
1) (TC) triangle condition: $\forall\, u, v, z$ s.t. $u \sim v$, $d(u,z) = d(v,z) = k$, $\exists\, w \sim u, v$ s.t. $d(w,z) = k-1$



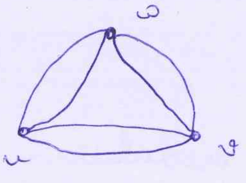
2) (QC) Quadrangle condition: $\exists\, w \sim v, v'$.



Pseudo-modularity: u, v, $d(u,v) \leq 2$, $d(u,z) = d(v,z) = k \Rightarrow \exists\, w$ s.t. $w \sim u, v$, $d(w,z) = k-1$.

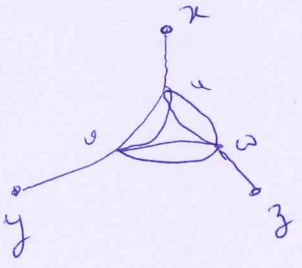


(u, v, w) is a <u>metric triangle</u> if $I(u,v) \cap I(u,w) = \{u\}$
$I(v,u) \cap I(v,w) = \{v\}$
$I(w,u) \cap I(w,u) = \{w\}$



Quasi-median of x, y, z : Metric triangle $(u, v\, w)$ such that

1) $d(x,y) = d(x,u) + d(u,v) + d(v,y)$
2) $d(y,z) = d(y,v) + d(v,w) + d(w,z)$
3) $d(z,x) = d(z,w) + d(w,u) + d(u,x)$.
4) $d(u,v) = d(u,w) = d(v,w)$

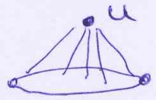


<u>Lemma 4</u> : $\forall$ graph G, $\forall$ triplet x, y, z, $\exists$ at least one quasi-median.

<u>Prop</u>: G is weakly-modular $\Leftrightarrow$ $\forall$ metric triangle $(u\, v\, w)$,

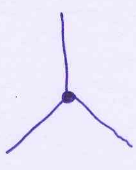
$\forall\ x, x' \in I(v,w)$, $d(u,x) = d(u,x')$.

In weakly modular graphs, metric triangles are strongly equilateral.

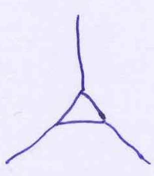


Modular graphs $\equiv$ graphs where all metric triangles are vertices.

<u>Lemma 5</u> : Pseudomodular $\Rightarrow$ metric Δles have size 0 or 1.



or



~~Proof~~:

Lemma 6: If G is pseudo-modular $\Rightarrow$ any three pairwise intersecting balls intersect.

Lemma 7: Helly $\Rightarrow$ pseudo-modular $\Rightarrow$ weakly modular.

Sketch of proof of Thm 1