Chepoi Lecture 4

16/5/2019

Prop 1 CAT(0) cubical groups are Helly.

Prop 2 (Lang) Hyperbolic groups are Helly.

Recall: Any graph G embeds in its injective hull $E(G)$ via the Kuratowski map $G \xrightarrow{e} \underset{\text{external}}{E(G)} = \{ f \in \mathbb{R}^G : f(x) + f(y) \geq d(x,y)\ \forall x, y \in G \}$

Proof: Let $\Gamma \curvearrowright G$, hyperbolic graph.

(Chepoi)

1) $\forall g \in \Gamma$, $g : G \to G$ can be extended to an isometry
$g : E(G) \to E(G)$

2) $E(G)$ is proper if the intervals of G are β-stable. β-stable : $\forall$ triple $x, y, z \in G$ with $d(x,y) = 1$, the Hausdorff distance between the intervals $I(x,z)$ and $I(y,z) \leq \beta$.

Eg: Weakly modular graphs are 2-stable.

3) Lemma 1: $\forall f \in E(G)$, $\exists x \in G$ s.t. $d_\infty(f, e(x)) \leq 2\delta$

4) Γ acts properly discontinuously on $E(G)$.

Let $K \subseteq E(G)$ be compact.

Consider $K' = \{ x \in G : \exists f \in K, d_\infty(f, e(x)) \leq 2\delta \}$

Let $g \in \Gamma$ be s.t. $gK \cap K \neq \phi$.

$K \cap gK \neq \phi \Rightarrow \exists f \in K$ s.t. $gf \in K$.

If $x \in G$ is s.t. $d_\infty(f, x) \leq 2\delta$, then $gx \in K'$. $\therefore K' \cap gK' \neq \phi$.

Since $|\{ g \in \Gamma \mid gK' \cap K' \neq \phi \}|$ is finite, $|\{ g \in \Gamma \mid gK \cap K \neq \phi \}|$ is finite.

5) Γ acts cocompactly :

Let R be s.t. $\cancel{E(G)}\ \Gamma \cdot B(x, R) = G$ for some $x \in G$.

Then $R' := R + 4\delta$ works for $E(G)$.

Coarse Helly property :

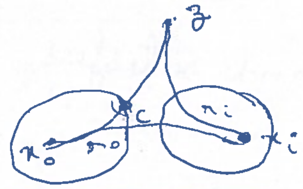
Lemma 2: G hyperbolic. Let $B(x_i, r_i)$ $i \in I$ be a family of balls such that $r_i + r_j \geq d(x_i, x_j)$. i.e $B(x_i, r_i) \cap B(x_j, r_j) \neq \phi$.

Then $\bigcap B(x_i, r_i + 2\delta) \neq \phi$.

Sketch: Let $z \in G$ be arbitrarily. Suppose $d(z, B_0) \geq d(z, B_1) \geq \dots$ (where $B_i = B(x_i, r_i)$). If $d(z, B_0) \leq 2\delta$, we are done.

otherwise, pick $c \in I(x_0, z) \cap B_0$ s.t. $d(x_0, c) = r_0$.

Then $\forall i \in I \setminus \{0\}$, $d(c, x_i) \leq r_i + 2\delta$.



Proof of Lemma 1 : Let $f \in E(G)$

Consider balls of G of radius $f(x)$, $x \in G$.

Since $f(x) + f(y) \geq d(x, y)$, by Lemma 2,

$$\exists\ c \in \bigcap_{x \in G} B(x, f(x) + 2\delta).$$

Claim: $d_\infty(f, e(c)) \leq 2\delta$.

$$d_\infty(f, e(c)) = \sup_x |f(x) - d(c, x)|$$

$$d(x, c) - f(x) \leq 2\delta \quad \text{, by Lemma 2.}$$

Also, $f(x) - d(x, c) \leq 2\delta$.

If not, $f(x) > d(x, c) + 2\delta \rightarrow (a)$

Since $f \in E(G)$, $\exists\, y$ s.t. $f(x) + f(y) = d(x, y)$.

But $f(y) \geq d(y, c) - 2\delta \rightarrow (b)$

Adding (a) and (b), $f(x) + f(y) > d(x, c) + d(y, c) \geq d(x, y)$. #

<u>Sweakly modular graphs</u>

"super weakly"

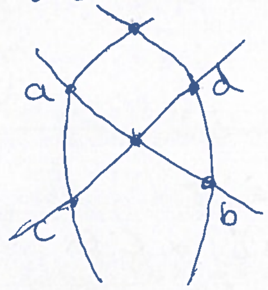
Point-line geometries $\Pi = (P, L)$ ~~sets of~~ tuple of "points" and "lines"

with the condition that two points are contained in at most one line.

$G = G(\Pi)$ collinearity graph. (vertices $= P$, p, q adjacent if $p, q \in \ell \in L$)

Subspace $S \subset P$ s.t. ~~if $s \in S$ & s', s collinear, $s' \in P$~~ $\forall \ell \in L$, either $\ell \subseteq S$ or $|\ell \cap S| \le 1$.

<u>Projective space</u> $\forall$ point $\exists$ a line passing through the point and satisfies the Veblen-Young axiom:

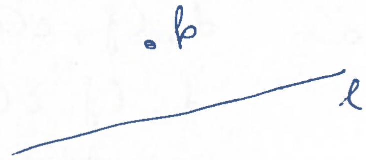


Dimension: max. chain of proper ~~p~~ subspaces.

<u>Polar space</u> (Tits, 74)

<u>Def 1</u>: $\forall$ ~~p~~ ℓ, p s.t. $p \notin \ell$,

either p is collinear with a unique point of ℓ or with all points of ℓ.



<u>Def 2</u>: 1) Any maximal proper subspace is a ~~maximal~~ projective space with subspaces.

2) intersection of subspaces is a subspace.

3) for any maximal proper subspace U and $p \notin U$, $\exists$ unique

max subspace $W \ni p$ s.t. $U \cap W$ is a subspace of dimension one less.

Dual polarity : $\Pi = (P, L)$ polar gives rise to a dual $\Pi^* = (P^*, L^*)$

P^* = all maximal proper $(n-1)$ dimensional subspaces of Π.

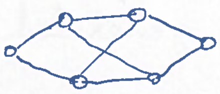
L^* = all $(n-2)$ dimensional subspaces of Π

Dual polar graph : Collinearity graph of a dual polar space.

Thm (CCHO) For a graph G, TFAE :

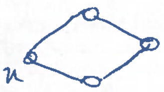
1) G is dual polar

2) G is a thick weakly modular graph without $K_4^- =$ 

and $K_{3,3}^- =$ 

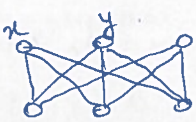
3) G is thick, is locally weakly modular, no K_4^-, $K_{3,3}^-$

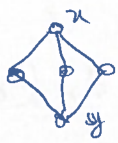
and $X_\Delta(G)$ is simply connected.

Thick : $d(x,y) = 2 \Rightarrow \exists$ square containing them : 

Sweakly modular graph : Graph G weakly modular, no K_4^-, $K_{3,3}^-$.

Boolean pair : (x, y) s.t. $I(x,y)$ contains a Boolean cube of dimension $d(x,y)$.

$K_{3,3}$ 

$I(x,y) =$ 

Prop: 1) (x,y) is a Boolean pair

2) $I(x,y)$ is a complemented modular lattice.

3) The graded hull $\langle\langle x,y\rangle\rangle$ is a dual polar space.

Thickening G^Δ of G: $x \sim y$ in G^Δ iff (x,y) is a Boolean pair

Thm: G^Δ is Helly.

G. swm graph. $(K(G), d_\infty)$ - injective

$(K(G), d_2)$ - CAT(0) (Hirai, 2019)

$K(G)$ contractible.

Thm (CCGHO) Helly groups are biautomatic.

Biautomatic structure of a group Γ, generated by S.

Language: $\mathcal{L} \subset (S \cup S^{-1})^*$

Regular language: recognized by finite automaton

Biautomatic structure: regular language $\mathcal{L} \subset (S \cup S^{-1})^*$ satisfying the 2-sided fellow traveler property in $\mathrm{Cay}(\Gamma, S)$

Fellow traveling:

P - set of paths of a graph G s.t.

1) complete $\forall (x,y)\ \exists\ \gamma(x,y) \in P$
$\gamma(y,x) \in P$

2) $\exists\ c > 0,\ D > 0$

$\forall \gamma_1, \gamma_2 \in P,\ \forall i \in \mathbb{N},\ d_G(\gamma_1(i), \gamma_2(i)) \le c \cdot \max\{d_G(\gamma_1(0), \gamma_2(0)), d_G(\gamma_1(\infty), \gamma_2(\infty)) + 1\}$

Świątkowski's approach

$\Gamma \underset{geom}{\curvearrowright} G$ graph $\gamma_1, \gamma_2 \in G$ congruent if $\exists g \in \Gamma$ s.t. $g\gamma_1 = \gamma_2$.

$\mathcal{S}_k$ - the Γ-congruence classes of paths of length k.

$\gamma \to [\gamma]$ - equivalence class.

Def: Let $R \subset \mathcal{S}_k$ and let $\mathcal{P}_k(R)$ be the path system of G consisting of all paths γ s.t.

(1) if $|\gamma| \geq k$, then $[\eta] \in R$ $\forall \eta$ subpath of length k of γ.

2) if $|\gamma| < k$, then γ is a prefix of a path γ' with $[\gamma'] \in R$.

A path system P of G is k-locally recognized if $\exists R \subset \mathcal{S}_k$ s.t.

$P = \mathcal{P}_k(R)$

Thm (Świątkowski): Let $\Gamma \curvearrowright G$ geometrically. If P is a path system of G s.t.

1) P is k-locally recognized for some k,

2) P is complete

3) P satisfies the 2-sided fellow traveller property

Then Γ is biautomatic

Normal clique paths in Helly graphs

$S \subset G$, $B^*(S, k) = \bigcap_{x \in S} B(x, k)$

Two cliques σ, τ of G are at uniform distance k if $d_G(s, t) = k$ $\forall s \in \sigma$, $\forall t \in \tau$. Notation $\tau \bowtie_k \sigma$

Let $\tau \bowtie_k \sigma$.

$$R_\tau(\sigma) = B_k^*(\tau) \cap B_1^*(\sigma)$$

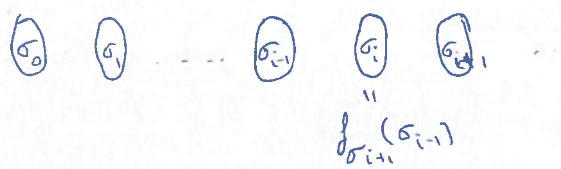
$$f_\tau(\sigma) = B_{k-1}^*(\tau) \cap B_1^*(R_\tau(\sigma))$$

<u>Lemma 1</u>: $f_\tau(\sigma)$ is non-empty, $f_\tau(\sigma) \cup \sigma$ is a clique and $f_\tau(\sigma) \bowtie_{k-1} \tau$, $f_\tau(\sigma) \bowtie_1 \sigma$.

($f_\tau(\sigma) \neq \phi$ by Helly property).

<u>Normal clique-path</u>: $(\sigma_0, \sigma_1, \dots, \sigma_k)$ s.t.

1) $\sigma_i \cap \sigma_{i+1} = \phi \ \forall i$
2) $\sigma_i \cup \sigma_{i+1}$ is a clique
3) $\sigma_{i-1} \bowtie_2 \sigma_{i+1} \ \forall i$
4) $\sigma_i = f_{\sigma_{i+1}}(\sigma_{i-1})$.



<u>Thm</u>: For any pair of cliques τ, σ s.t. $\tau \bowtie_k \sigma$, $\exists$ a unique <u>normal clique path</u> $\gamma_{\tau\sigma} = (\tau = \sigma_0, \sigma_1, \dots, \sigma_k = \sigma)$ whose cliques are defined by $\sigma_i = f_\tau(\sigma_{i+1})$, $\forall i = 1, \dots, k-1$.

<u>Proof</u>: <u>Step</u> 1: Prove that $\gamma_{\tau\sigma}$ is a normal clique path

<u>Step</u> 2. $\gamma_{\tau\sigma}$ is the unique normal clique path.

<u>Prop</u>: Suppose $d(x,y) = k$, $d(x',y') = k$, $d(x,x') = 1 = d(y,y')$. Then the normal clique-paths from x to y & from x' to y' fellow-travel.