3-MANIFOLD COHERENCE

Thm [Scott/Sholen 73]

If $\pi_1 M^3$ is f.g then $\pi_1 M^3$ is f.p.

Corollary $\pi_1 M^3$ is coherent

thm [Scott] M^3 has compact core property.

Def. G is indecomposable if $G \not\cong A * B$ for $A, B \neq 1$

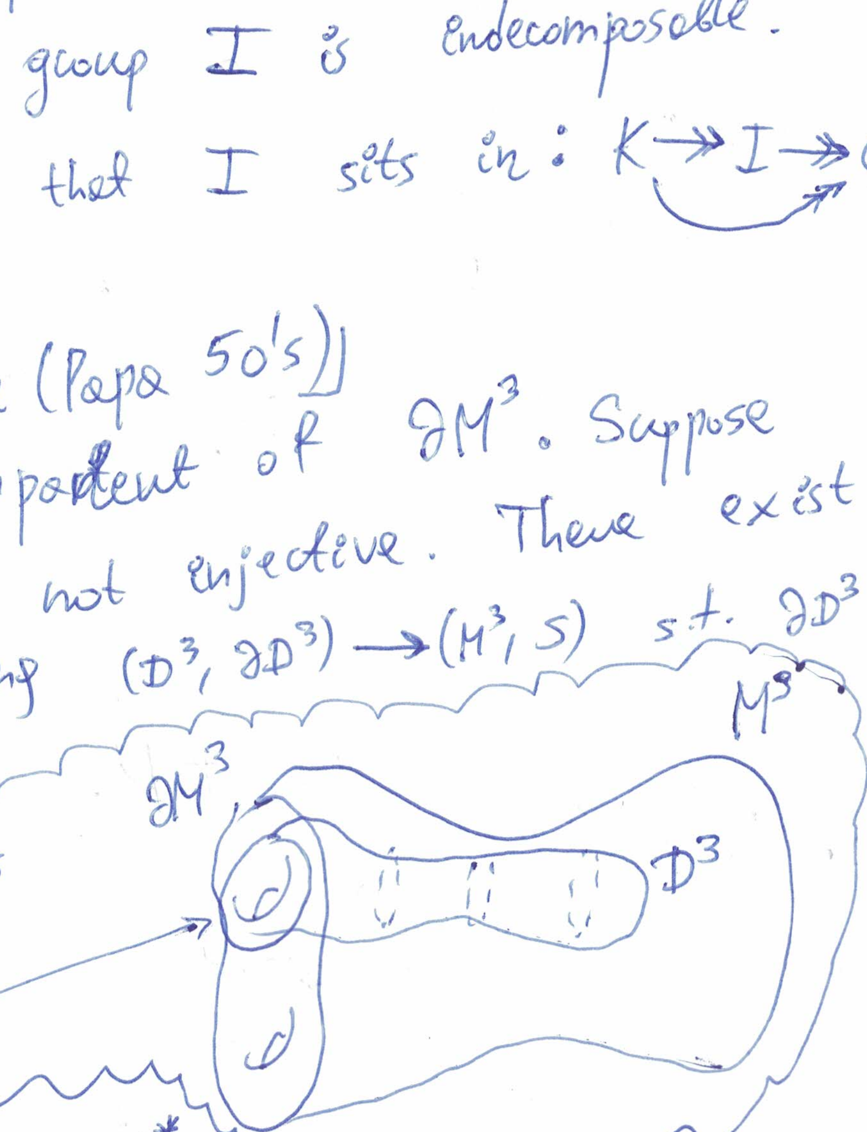
Scott's Lemma. Let G be f.g. indecomposable group. Suppose each subgroup $S \subseteq G$ with $rk(S) < rk(G)$ is f.p. Then $\exists$ f.p. K and $K \twoheadrightarrow G$ s.t. any intermediate group I is indecomposable.

[Intermediate means that I sits in: $K \twoheadrightarrow I \twoheadrightarrow G$].

Thm [Loop + Disk (Papa 50's)]

Let S be component of ∂M^3. Suppose $\pi_1 S \to \pi_1 M^3$ is not injective. There exist a proper embedding $(D^3, \partial D^3) \to (M^3, S)$ s.t. ∂D^3 is essential in S.

suppos this loop is in $\ker(\pi_1 S \to \pi_1 M^3)$



Proof of f.p. of $\pi_1 M^3$. Induction on $rk(\pi_1 M^3)$

True when $rk = 0$ or $rk = 1$. Suppose $\pi_1 M = A * B$ then $rk A, rk B < rk(\pi_1 M)$ by Grushko theorem.

Hence A and B are f.p. hence $\pi_1 M^3$ is f.p.

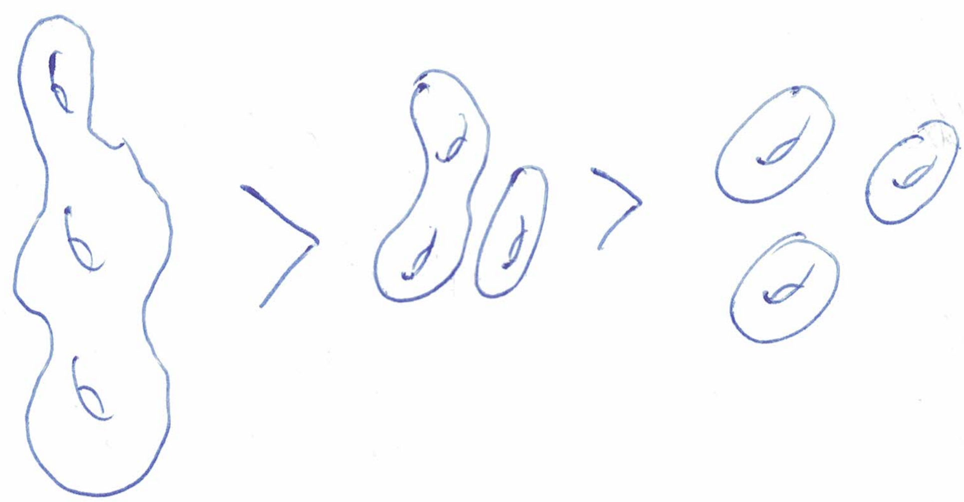
Using Scott's Lemma, there is f.p. group K and $\delta : K \twoheadrightarrow \pi_1 M^3$ with indecomposable intermediate

Let $N \subseteq M^3$ be compact submanifold s.t. $\pi_1 N$ contains image of $X \to M$ where $K = \pi_1 X$ and s.t. map $X \to M$ corresponds to $K \to \pi_1 M$.

Complexity of ∂N is minimal, where complexity is defined to be: $(\dots, c_2, c_1, c_0)$ where

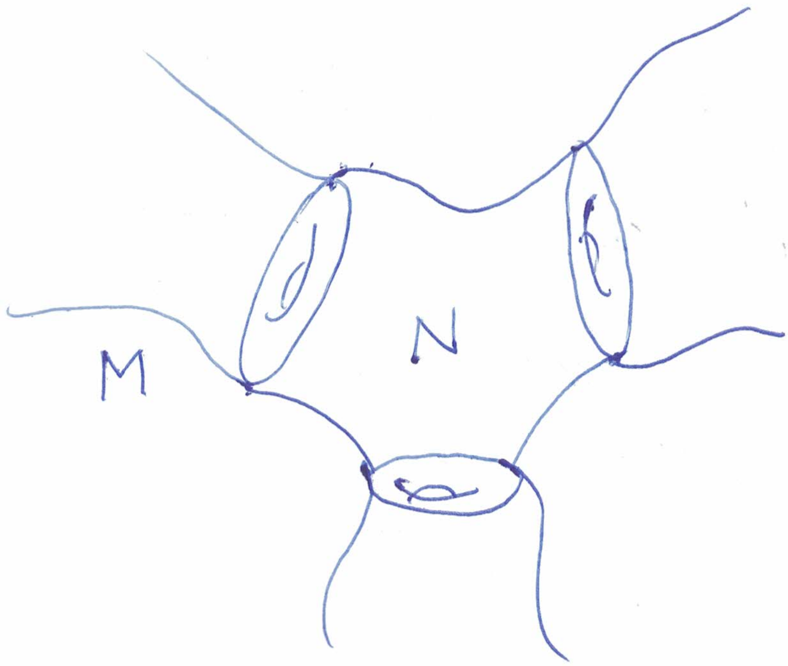
$c_i :=$ # components of ∂ with genus i.

For example:



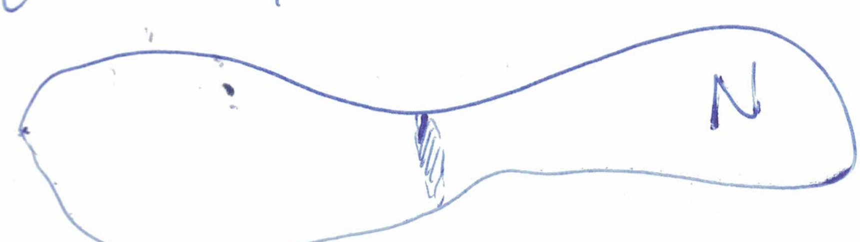
Claim. $\pi_1 N$ injects to $\pi_1 M$.

Proof. Suppose $\partial N \to M$ is π_1-injective on each component. Then $\pi_1 N \to \pi_1 M$ is injective.



Indeed we obtain graph of spaces with injective edge spaces. Hence vertex groups injects in graph of groups.

Assume component S of ∂N has $S \to M$ and not π_1-injective. By loop-disk theorem there exist compression disk D^2 with $\partial D^2 = S^1$. If D^2 is outside N then cut-paste gives lower complexity $N' = N + D^2 \times [0,1] - \partial D^2 \times [0,1]$. Then take S' to be boundary component of N'. And we have $S' < S$ which is not possible. If D^2 is inside N, then N splits along D^2.



Thm [Grushko] Let $\varphi: F \twoheadrightarrow A*B$ where F is f.g. free. Then $F = A' * B'$ with: $\varphi(A') = A$, $\varphi(B') = B$.

Corollary. If $G = A*B$ then $rk(G) = rk(A) + rk(B)$

Strong Grushko: Let $G = G_1 * \cdots * G_r * F_s$ and

F_s: free

$G_i \ncong \mathbb{Z}$ and indecomposable.

suppose $G_i \not\cong \mathbb{Z}$ and indecomposable.

Let $\varphi: G \twoheadrightarrow H = H_1 * \cdots * H_\ell$ and $\varphi(G_i) \subseteq H_i$

there exists decomposition $G = K_1 * K_2 * \cdots * K_\ell$

s.t. $\varphi(K_i) = H_i$

Complexity$(G) := (r+s, s)$, where $G = G_1 * \cdots * G_r * F_s$.

G: f.g

$G_1, \dots, G_r$: non $\mathbb{Z}$ indecomposables

Idea: higher complexity $\rightsquigarrow$ closest to $F_{rk(G)}$

Def. $\varphi: G \to \overline{G}$ is semi-injective if injective in each non $\mathbb{Z}$ free factor.

Thm If $G \xrightarrow{\varphi} \overline{G}$ is noninjective epimorphism then:

$$\text{Complexity}(G) > \text{Complexity}(\overline{G}).$$

Sketch of proof of Scott's lemma: Let G be f.g indecomposable ~~with~~ as in statement.

Among all epimorphisms $K \twoheadrightarrow G$ with $rk(K) = rk(G)$ and K f.p. and $K \to G$: semiinjective choose K s.t. complexity of K is maximal.

Suppose $\exists$ intermediate: $K \twoheadrightarrow B_1 * B_2 \overset{\iota}{\twoheadrightarrow} G$

Let $C_i := \iota(B_i)$. then $rk(C_i) < rk(G)$, so C_i is f.p. by hypothesis. Hence either

$K \to C_1 * C_2$ is isomorphism or complexity of $C_1 * C_2$ is less than K $\blacksquare$