John McCammond course   30 N 2019

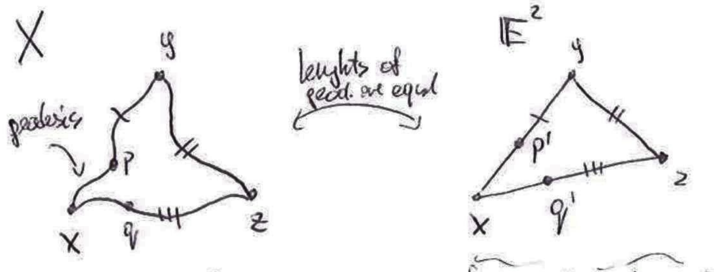
Topics: Non-positive curvature & braid groups

Section: Basics and buildings

Excersise (advanced?)
What is the appropriate notion of an „angle" between two planes intersecting in some Euclidian space?
(parameter space for 2×2-planes)

We want to encourage insight to basic principles, like in the excersise.
Understanding basic geometry is really interesting.

Definition. Let X be a geodesic metric space. A space is CAT(0) if all geodesic triangles are „at most as thick" as Euclidean triangles.



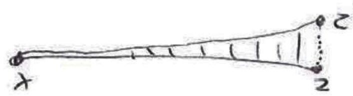
{comparison triangle in $\mathbb{E}^2$}

lenghts of these geodesics should at least satisfy triangle inequality

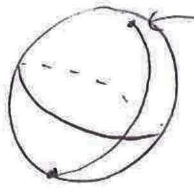
For every choice of $x, y, z, q, p, \ldots$

$$d_X(p,q) \leq d_{\mathbb{E}^2}(p',q')$$

But this characterisation is a bit hard to check. But it has a nice consequence: uniqueness of geodesics. (excercise)
Also: stability of geodesics: small changes to endpoints result in small variation of geodesics:



this does not happen on the sphere:



very close to north pole.
If moved to ~~n~~ ... — we have ambiguity and „jumps".

Further consequence: CAT(0) ⇒ contractible

Definition is hard to check. How do we know a CAT(0) space when we see one?

Examples

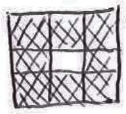
0) Euclidian spaces
1) Hyperbolic spaces – triangles are thin
⇒ 2) Universal covers of non-positively curved Riemmanian mflds.
This explains why S^2 is a non-example - can't put a non-positive metric on it (by Gauss-Bonnet for example)

Various notions of curvature in higher dimensional mflds were complicated, we discovered that the triangle condition is sufficient for contractibility and uniqueness of geod.

Cartan
Alexandroff
Top
+
Hadamard

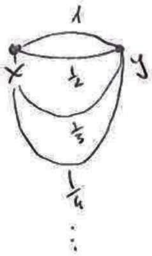
Definition A space is locally CAT(0) if $\forall x \in X$ $\exists$ nbhd $U \ni x$ s.t. U is CAT(0)

Theorem (Cartan – Hadamard)

X – locally CAT(0) and complete $\Rightarrow \tilde{X}$ is CAT(0)   **Example**  $= X$

Fun facts $D^2 \setminus \{0\}$ – its universal cover – „infinite parking garage" has bounded diameter – check

Idea Restrict the spaces we consider to spaces built of „nice" CAT(0) pieces, for examples convex hulls of points in $\mathbb{R}^n$, conv. polytopes

Non-example 

„Metric graph". Nice properties – semi-locally simply connected, uni-cover is a tree $\to$ CAT(0), but there is no path realising the infimum of distances.

Theorem X is piecewise euclidean (PE) cell complex built out of Euclidean polytopes glued together by faces* and there are only finitely many isometry types of faces then X is a geodesic metric space.

Remark Authors of Thm also describe CAT(1) spaces – where comparison $\triangle$ lie on S^2.

Checking if space is CAT(0) $\iff$ checking if links are CAT(1)

Key thing for CAT(1): does there exist a $\underset{<2\pi}{\text{short}}$ closed geodesic loop

Section: Crash course on buildings

$SL_3 / SO(3)$

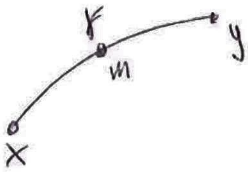
Definition A Riemm. mfld M is homogenous if $\mathrm{Isom}(M)$ acts transitively on points of M.

This gives some symmetry – every point „looks the same"

Definition A Riemm mfld M is a symmetric space if $\forall x \in M$ $\exists$ isometric of M fixing x and acting on T_xM by the antipodal map.

Example S^2

Symmetric $\Rightarrow$ homogenous



γ-geodesic, m-midpoint. Then reflection in m gives the isometry sending x to y, y to x. Careful, check for technicalities.

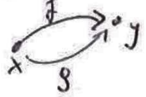
Surprising: Symmetric $\Rightarrow$ Isometry group has a Lie group structure!

Sym + compact $\Rightarrow$ Isometry group is compact

$\underset{\text{fixing } x}{\mathrm{Isom}_x(M)} = \underset{\text{compact Lie subgroup}}{K} \subset \mathrm{Isom}(M)$

Also M can be „reconstructed" from $K \subset G$

Remark $\underset{\mathrm{Isom}_x(M)}{K}$ fixes x. Let $\underset{\mathrm{Isom}(M)}{f, g}$ be s.t. $f(x) = g(x) = y$



$g^{-1}f \in K \to fK = gK$

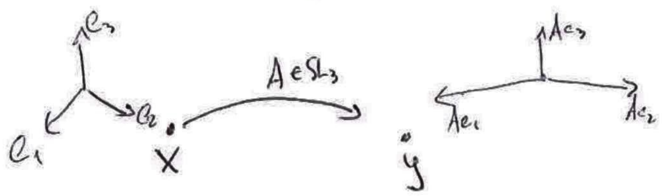
so points of M correspond to cosets of K

$G/K \overset{\text{bijects}}{\longleftrightarrow} M$. Dig into Haar measures to reconstruct M.

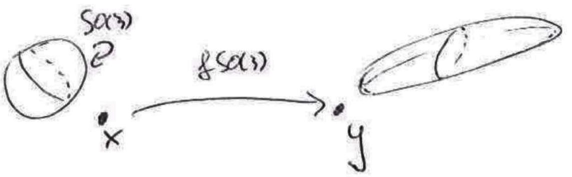
Back to $\underbrace{\underset{\substack{8\text{ dim}\\ \text{mfd}}}{Sl_3} / \underset{\substack{3\text{dim}\\ \text{mfd}}}{SO(3)}}_{\substack{\text{quotient is 5-dim}\\ \text{mfd}}}$. Consider the map $\det: M_{3\times3} \to \mathbb{R}$.

(actually with non-positive curvature). But the „flat" parts are highly symmetric

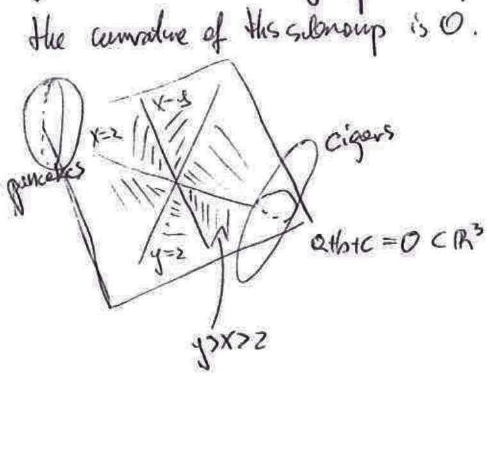
Points of Sl_3: for any $x \in Sl_3$ label it with a standard 3-basis of $\mathbb{R}^3$:



<u>Quotient</u> Replace unit vectors with sth $SO(3)$-



We have a Cartan subgroup in Sl_3: $\begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \end{bmatrix}$, trace $=0$. Exponentiate that to $\begin{bmatrix} e^a & & \\ & e^b & \\ & & e^c \end{bmatrix}$ - det $=0$. Check that the curvature of this subgroup is 0. Cartan subalgebra?



Coxeter complex for symmetric groups!

There exists an interesting structure at infinity

Excersise answer

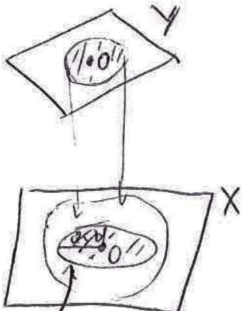
Parametrize two 2-planes in $\mathbb{R}^4$ to describe the „angle" between them. More generally, 2 k-planes in $\mathbb{R}^{2k}$

Reduction:

X^k, Y^l — linear subspaces of same $V \underset{WLOG}{=} \mathrm{Span}(X \cup Y)$

if $\dim(X \cap Y) > 0$ look at orthogonal complements and their intersections, and reduce to $\mathbb{R}^{2k}$

Idea: Restrict to a radius 1 ball in Y and X: project the unit ball in Y to X and analyze the image.



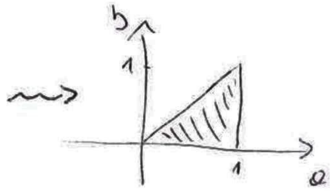
ellipsoid

From there you can reconstruct the „angle". Notice that we could've projected X to Y, but then the image of the unit ball of X in Y is just the same.

$1 > a \geq b > 0$ – the axes of the ellipsoid

sharp bc of the trivial intersections

$\rightsquigarrow$



this is the parameter space we wanted.

This generalizes to parameter space of $\mathbb{R}^{2k}$ case as follows: take the convex hull of points $(0,\dots,0), (1,0,\dots,0), (1,1,0,\dots,0), \dots, (1,1,\dots,1)$ – orthoscheme

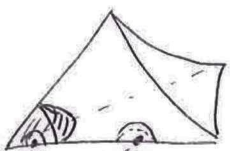
Coxeter calls the orthoscheme a convex hull of a pairwise orthogonal tuple of vectors.

Definition CAT(0)

Theorem (C-H) X - complete geodesic metric space. Then X is CAT(0) $\iff$ ① X is locally CAT(0) ② every closed piecewise geodesic is monotonically shrinkable to a constant path

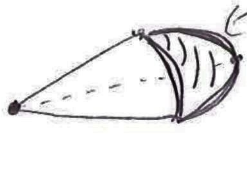
Reduction

For PE complexes with finitely many shapes completness and geometric met. sp are automatic.
It is sufficient to check CAT(0) for vertex nbhds:



copy of nbhds of points from the whole space inside nbhs of vertices

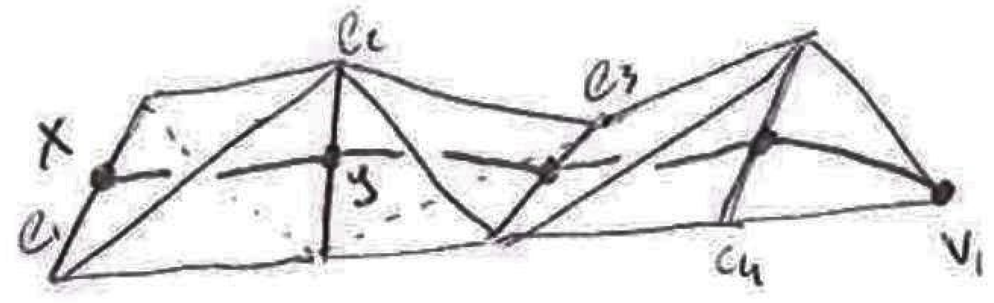
But then again it is enough to show CAT(1) for ~~rescaled~~ rescaled vertex link:

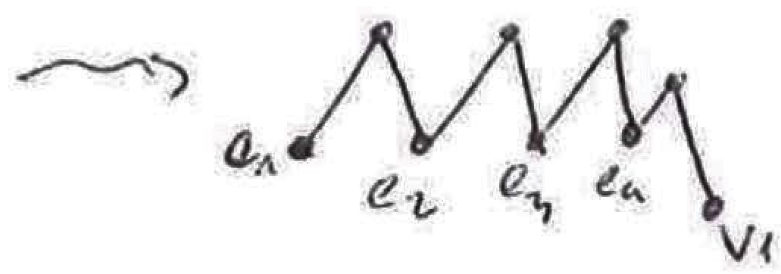


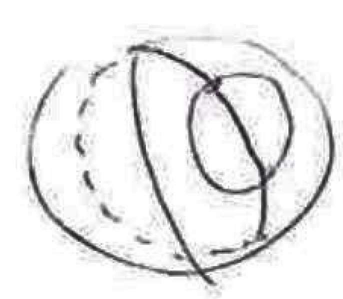
is this CAT(1) ?

A piecewise spherical complex Y is CAT(1) $\iff$ ① Y is locally CAT(1) and ② Every piecewise geodesic loop of lenght $< 2\pi$ is monotonically shrinkable to a constant path.
This is a result of Bowditch

Using face lattices to describe geodesics (precise geodesics)

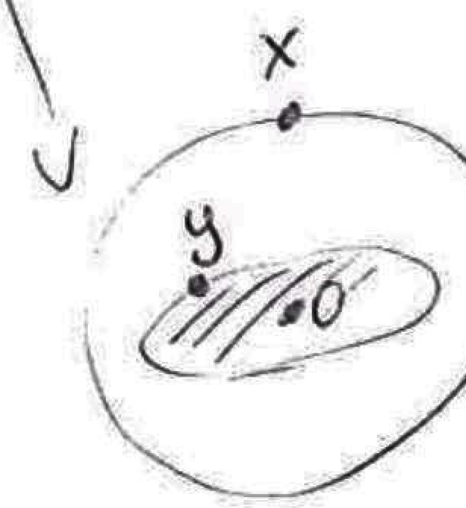




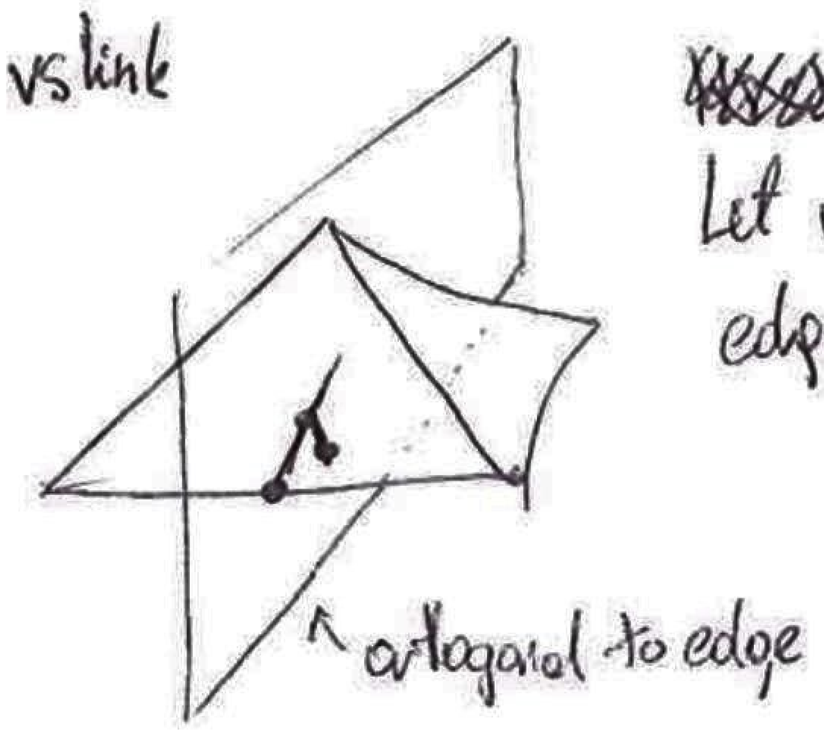


Every edge describes some great circle on some sphere - that determines a plane in $\mathbb{R}^{n+1}$. Given two edges one can calculate
of dimn

the angle between them using the excercise. Now minimize the ~~d~~ lenghts of curves by analyzing the seial unit disk projections.
(between the planes determined by them)



Link vs link



~~[illegible]~~
Let us look at the intersection of a „lemon wedge" Link with the orthogonal plane to an edge and call that the lowercase link.

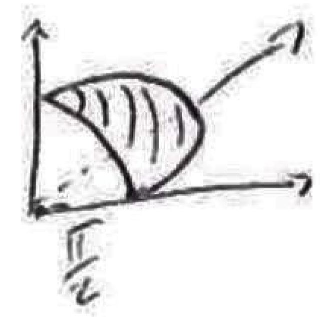
Using spherical joins: $\mathrm{Link}(x,K) = S^{k-1} * \mathrm{link}(x,K)$ $x \in K$

lowercase link

X is locally CAT(0) iff every link of every every face has no unshrinkable short geodesics.

Suppose X is a cube complex.

link (v,X) is P.S.



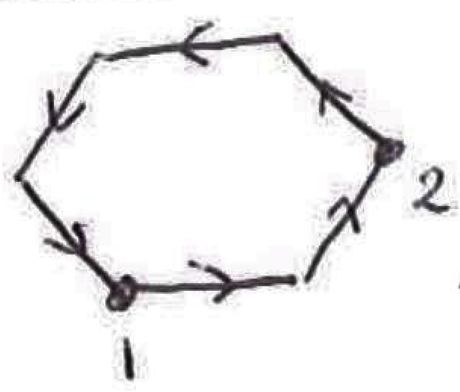
Theorem (Gromov)

A P.E. cube complex is CAT(0) $\Longleftrightarrow$ every vertex link is a flag complex

Theorem (Moussong)

If K is a PE complex in which every dihedral angle is $\geq \frac{\pi}{2}$ is CAT(0) $\Longleftrightarrow$ every vertex link is metric flag

Excercise



*~~Configuration graph for motions of robots~~ 1,2, where lenght of edge is determined by the number of robots moving. Show it's locally CAT(0)

*Consider the graph for configuration space of two robots on vertices of a hexagon (pairs of distinct vertices), where two vertices are connected if a point is moved to a neighbouring vertex.

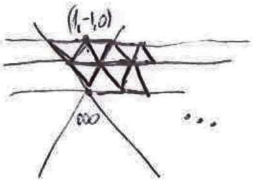
Definition Ordinary braid group

⋮

Consider the action of $S_3 \curvearrowright \mathbb{R}^3$ (permuting the coordinates).

Also look at the reflection $(x,y,z) \longmapsto (y+1, x-1, z)$ which moves the origin to $(1,-1,0)$.

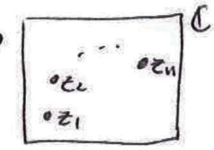
Now this tiles the $x+y+z=0$ plane into triangles (restricting the action to the plane)



Braids

Look at $\mathbb{C}^n \setminus \mathcal{H} = \{\text{hyperplane } x_i = x_j\}$

$(z_1, \dots, z_n) \rightsquigarrow$



If we move our point in $\mathbb{C}^n$ we obtain a certain pattern down below. We'll be interested in loops to get fundamental group.

$\pi_1(\mathbb{C}^n \setminus \mathcal{H}) =$ pure braid group PBraid_n

S_n acts freely, discontinuously, ..., on $\mathbb{C}^n \setminus \mathcal{H}$. So we can factor:

$\pi_1\left(\mathbb{C}^n \setminus \mathcal{H} / \text{Sym}_n\right) =$ braid group Braid_n

We have an exact sequence

$$\text{PBraid}_n \hookrightarrow \text{Braid}_n \twoheadrightarrow \text{Sym}_n$$

Braid group $:= \pi_1(\text{UConf}_n(\mathbb{C}), z_0)$ ← some base point

↑ unordered configuration space

Example $\text{Braid}_3 = \langle a, b \mid aba = bab \rangle$

In general $\sigma_i = (i, i+1)$

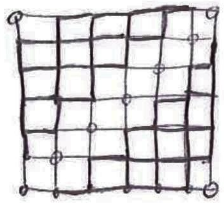
$$\text{Sym}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i^2 = 1 & \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| > 1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & |i-j| = 1 \quad (3) \end{array} \right\rangle$$

Braid_n is the same but omit $\sigma_i^2 = 1$.

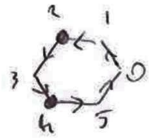
Sometimes people write diagrams for these presentations; label vertices by generators and connect them if they satisfy some nonobvious relation e.g.

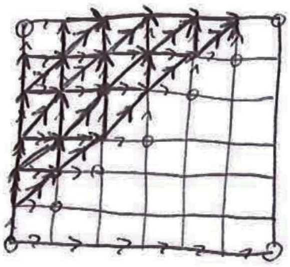
$$\text{Braid}_3 = a \overset{(3)}{\text{———}} b$$

$\text{Conf}_2(\bigcirc) = (\bigcirc \times \bigcirc) \setminus$ thick diagonal

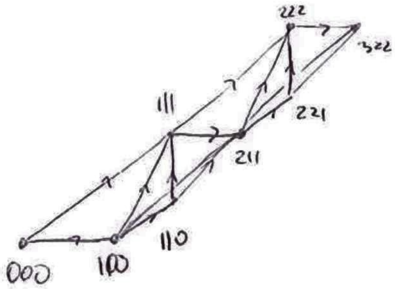


Z satisfies Gromov link condition



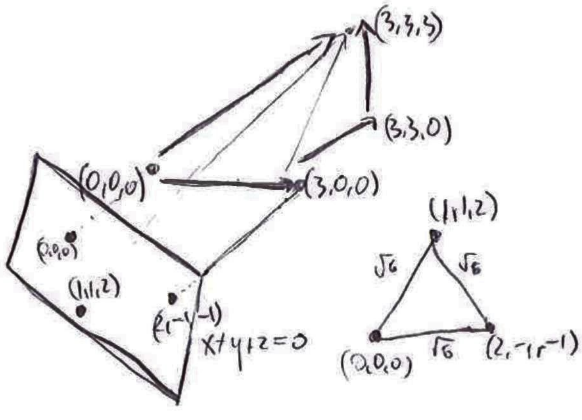


In n dimensions instead of triangles you obtain orthoschemes. For example $n=3$

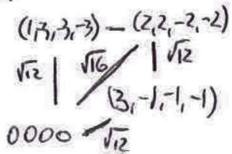


Such columns are invariant under translation ~~by~~ along the $x=y=z$ axis.

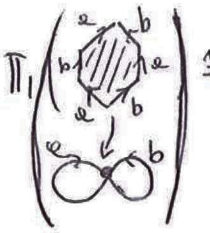
Tile $\mathbb{R}^3$ with cubes partitioned into orthoschemes. This in turn tiles the $x+y+z=0$ plane into equilateral triangles.



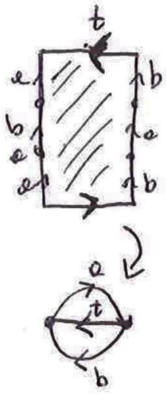
Similar projection of 4-dim orthoscheme gives

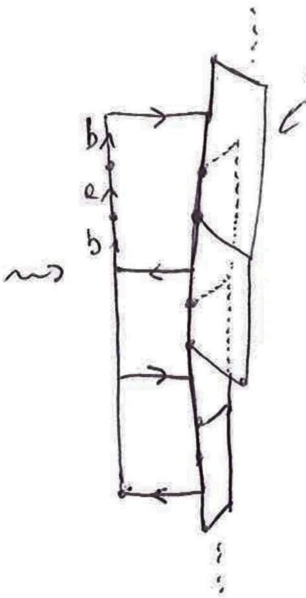


$\hat{A}_3$ Coxeter shape

π_1  $\cong$ Braid$_3$

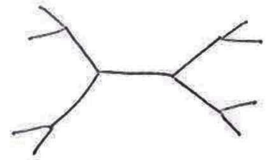
Universal cover? If we want to focus on the geometry of such objects, we need to change our algorithm of constructing the universal cover.



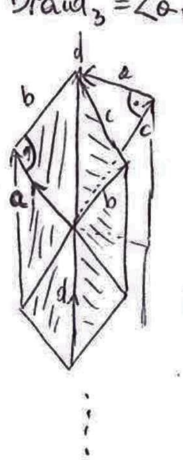


Sorry, these should all be rectangles...

Cross section: $T_3 \times \mathbb{R}$

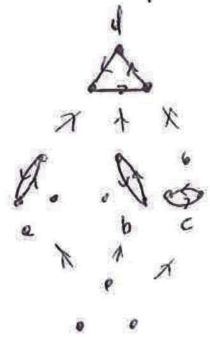


Braid$_3 = \langle a,b,c,d \mid ab = bc = ca = d \rangle$



$T_3 \times \mathbb{R}$

Find all partitions of a convex n-gon, no intersections:



That determines a set of generators: every such partition would be a generator. Take the directed Cayley graph with that generating set, and make it a flag complex.
(the condition of disjointness gives us commuting generators)

We have a natural homomorphism Braid$_n \twoheadrightarrow \mathbb{Z}$. So in this complex there are no cycles:

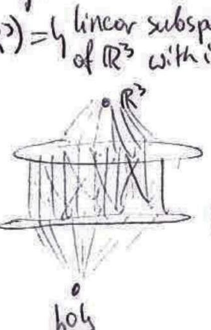
<u>Complex</u>
is the Cayley graph of Braid$_n$ with resp. to the non-crossing partition generators. Flag $\Rightarrow$ filled with orthoschemes

Graded posets $\longrightarrow$ orthoscheme Δ-complex

 $\rightsquigarrow$ metric complex - orthoscheme metric

<u>Fact</u> Orthoscheme metrics gives us isometry:
$|P \otimes Q| \cong |P| \times |Q|$
where the order on $P \otimes Q$ is $(p,q) \leq (p',q')$ iff $p \leq q$, $p' \leq q$

Buildings
$Lin(\mathbb{R}^3) = \{$linear subspaces of $\mathbb{R}^3$ with inclusion$\}$



planes thru 0
lines thru 0

$\leftarrow$ turn this into a Δ-complex with the orthoscheme metric

Building at ∞ of $SL_3(\mathbb{R})/SO(3)$ looks like geometric realization of $Lin(\mathbb{R}^3)$