Acylindrically Hyperbolic Groups

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Lecture 1

Let $\underset{\text{group}}{G} \curvearrowright \underset{\text{Metric space}}{S}$ by isometries. This action is

- <u>proper</u> if $\forall\, r > 0$, $\forall\, s \in S$, $|\{g \in G \mid d_S(s, gs) \le r\}| < \infty$
- <u>cobounded</u> if $\exists$ a bounded $B \subseteq S$ s.t. $S = \bigcup_{g \in G} gB$.
- <u>geometric</u> if it is proper & cobounded.

A map $f : (S, d_S) \to (T, d_T)$ is a quasi-isometry if $\exists\, C > 0$ s.t.

1) $\frac{1}{C} d_S(x,y) - C \le d_T(f(x), f(y)) \le C d_S(x,y) + C$ and

2) $f(S)$ is C-dense in T.

<u>Eg</u>: - G is f.g., X, Y are finite generating sets.

Then $(G, d_X) \underset{q.i.}{\sim} (G, d_Y)$.

- G is f.g., $H \underset{f.i.}{\le} G \Rightarrow H$ is f.g. and $H \underset{q.i.}{\sim} G$.

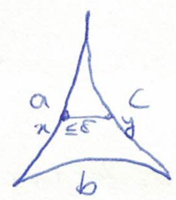
<u>Thm (Efremovich)</u> <u>Svarc-Milnor Lemma</u>. Suppose G acts coboundedly on a geodesic metric space S. Then $\exists$ a generating set $X \subseteq G$ s.t. the Cayley graph $\Gamma(G, X) \underset{q.i.}{\sim} S$. If, in addition, $G \curvearrowright S$ is proper, then $|X| < \infty$.

Eg: $M \to$ compact manifold. Then $\pi_1(M) \sim_{q.i.} \widetilde{M}$.

Hyperbolic spaces

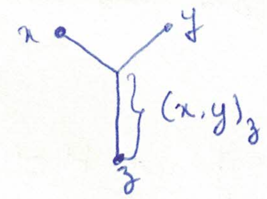
Def: A metric space S ~~of~~ is hyperbolic if

1) S is geodesic
2) $\exists\, \delta > 0$ s.t. $\forall$ geodesic Δ with sides a, b, c, $\forall\, x \in a$, $\exists\, y \in b \cup c$ s.t. $d_S(x,y) \le \delta$.



Eg: 1) Bounded spaces ($\delta = \operatorname{diam}(S)$)
2) $\mathbb{H}^n$.
3) Trees ($\delta = 0$).
4) $\mathbb{R}^n$ is not hyperbolic for $n \geq 2$.

Def: Given $x, y, z \in S$, the Gromov product of x, y wrt z is $(x,y)_z = \frac{1}{2}\left(d(x,z) + d(y,z) - d(x,y)\right)$



Homework #1: If S is a tree, then $\forall\, x, y, z, t \in S$,

$$(x,z)_t \geq \min\{(x,y)_t, (y,z)_t\}.$$

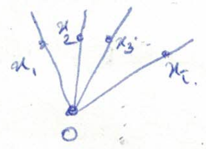
H.W. #2.
Thm 1: A m.s. S is hyp $\iff \exists\, \delta > 0$ s.t. $\forall\, x, y, z, t \in S$ we have $(x,z)_t \geq \min\{(x,y)_t, (y,z)_t\} - \delta$.

Gromov boundary

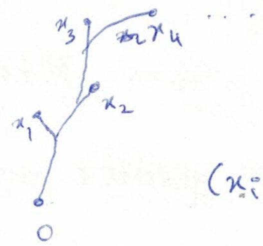
<u>Def</u>: Let S be hyperbolic. Fix $o \in S$.

A sequence $(x_i) \subseteq S$ <u>converges to ∞</u> if

$$\liminf_{i,j\to\infty} (x_i, x_j)_o = \infty.$$



$(x_i) \not\to \infty$



$(x_i) \longrightarrow \infty$.

Say $(x_i) \sim (y_i)$ if $\liminf_{i,j\to\infty} (x_i, y_j)_o = \infty$.

<u>H.W. #3</u>:

$\sim$ is an equivalence relation on the set of ~~equ~~ sequences $\to \infty$.

(transitivity by Thm 1).

<u>Def</u>: $\partial_o S := \{ [(x_i)] \mid (x_i) \to \infty \}$.

The base of neighbourhoods of $a \in \partial_o S$ is

$$\underset{r>0}{\mathcal{U}_a(r)} = \{ b \in \partial_o S \mid \exists\, (x_i) \in a, (y_i) \in b \text{ s.t. } \liminf_{i,j\to\infty} (x_i, y_j) \geq r \}$$

<u>Eg</u>: 1) $\partial_o(\text{bounded space}) = \phi$.

2) $\partial_o \mathbb{H}^n \cong \mathbb{S}^{n-1}$

3) S locally finite homogeneous tree, then $\partial_o S \cong$ Cantor set

4) $\partial_o \Big($  countably many ∞-rays $\Big)$ = discrete (∞) set.

<u>Prob</u>: $\partial T_\infty = ?$, where T_∞ is a tree where every vertex has countably infinite degree.

Thm (Gromov) If S is hyp, then

a) $\partial_o S$ is independent of the choice of o upto homeomorphism.

b) ∂S is a completely metrizable (Hausdorff) space. If S is proper, then ∂S is compact.

c) If R is geodesic and $R \sim_{q.i.} S$, then $\partial R \cong \partial S$.

d) If $G \curvearrowright S$, then this action induces $G \overset{\text{continuous}}{\curvearrowright} \partial S$, by

$$g : [(x_i)] \longmapsto [(g x_i)].$$

Group actions on hyp. spaces

Def: An element $g \in G \curvearrowright \underset{\text{hyp}}{S}$ is

- elliptic if $\langle g \rangle$ has bounded orbits.
- parabolic if $\langle g \rangle$ has unbounded orbits & fixes exactly 1 point on ∂S.
- loxodromic if $\langle g \rangle$ has unbounded orbits & fixes exactly 2 points on ∂S.

Thm: (Gromov): Every isometry of a hyp. space is elliptic, parabolic or loxodromic.

Eg: ~~$SL_2(\mathbb{Z})$~~ $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$

$|tr. A| < 2$ or $A = \pm I$, then A is ell.
$|tr. A| = 2$, A is par.
$|tr. A| > 2$, A is lox.

Let $G \curvearrowright S$ hyp. Given $R \subseteq S$, define

$\partial R \subseteq \partial S$, $\partial R = \{ [(x_i)] \in \partial S \mid (x_i) \subseteq R \}$.

<u>Def</u>: Fix $o \in S$. Let $G \cdot o = \{ go \mid g \in G \}$.

The limit set of G is $\Lambda(G) = \partial G \cdot o$.

<u>Rmk</u>: $\Lambda(G)$ is independent of o.

<u>Thm</u>. $\forall\, G \curvearrowright S_{hyp}$, $|\Lambda(G)| \in \{0, 1, 2, \infty\}$.

<u>Def</u>: The action $G \curvearrowright S$ hyp. is

- <u>elliptic</u> if $\Lambda(G) = \phi$ ($\Leftrightarrow$ orbits are bounded).
- <u>parabolic</u> if $|\Lambda(G)| = 1$.
- <u>lineal</u> if $|\Lambda(G)| = 2$
- <u>quasi-parabolic</u> if $|\Lambda(G)| = \infty$ & G fixes a point of ∂S.
- <u>of general type</u> if $|\Lambda(G)| = \infty$ & G does not fix any point of ∂S.

<u>Rmk</u>: $g \in G$ is ell. (resp., par, lox) if $\langle g \rangle$ is ell (resp., par, lineal).

Action	$\lvert\Lambda(G)\rvert$	$\mathrm{Fix}_G(\partial S)$	orbits	Types of elements
elliptic	0	any	bounded	elliptic
parabolic	1	1	unbdd, never quasiconvex	elliptic or parabolic
lineal	2	2	quasiconvex $\sim$ q.i. to $\mathbb{R}$	elliptic or loxodromic
quasiparabolic	∞	1 $\xRightarrow{\text{H.W.}}$	unbounded, q.c.	any $\{g^{\pm\infty} / g \text{ lox}\}$ is dense in $\Lambda(G)$.
gen type	∞	0	unbounded	any. $\{(g^{+\infty}, g^{-\infty}) \mid g \text{ lox}\}$ is dense in $\Lambda(G) \times \Lambda(G)$

gen. type	∞	0	unbounded	any. $\{(g^{+\infty}, g^{-\infty}) \mid g$ loxodromic$\}$ is dense in $\Lambda(G) \times \Lambda(G)$

1) $g \in G$ loxodromic. $\{g^{+\infty}, g^{-\infty}\} = \Lambda(\langle g \rangle)$

2. If S is hyp., $R \subseteq S$ is quasiconvex if $\exists\, \sigma > 0$ s.t. $\forall\, x, y \in R$, any geodesic in S connecting x to y belongs to $N_\sigma(R)$.

Thm: S hyp., $R \subseteq S$ quasiconvex $\Rightarrow R \underset{q.i.}{\sim}$ hyp. space.

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a) $\{(g^{+\infty}, g^{-\infty}) \mid g$ loxodromic$\}$ dense in $\Lambda(G) \times \Lambda(G)$

$\Rightarrow G \geq F_2$.

b) $\{g^{\pm\infty} \mid g$ loxodromic$\}$ dense in $\Lambda(G)$

$\Rightarrow G \geq$ free semigroup (of rank 2).

Eg: $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$.

1) $SO_2(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$ is elliptic.

2) $\mathbb{Z} \cong \left\{ \begin{pmatrix} e^n & 0 \\ 0 & e^{-n} \end{pmatrix} \mid n \in \mathbb{Z} \right\}$ $\quad |tr(A)| = e^n + e^{-n} \geq 2$.

$\mathbb{Z} \curvearrowright \mathbb{H}^2$ is lineal.

3) $UT_2(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} r \in \mathbb{R} \right\} \curvearrowright \mathbb{H}^2$ is parabolic.

4) $ST_2(\mathbb{R}) = SL_2(\mathbb{R}) \cap T_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$.

$ST_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ is q. parabolic.

Not of gen. type as $ST_2(\mathbb{R})$ is solvable ($\therefore \not\geq F_2$).

5) $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ is of general type.

Eg: $G \curvearrowright$ tree. Then every element is ell. or lox.

More generally, a f.g. gp G cannot act parabolically on a tree.

Eg: $G = A *_C B \curvearrowright$ Bass-Serre tree.

The action is

- elliptic $\Leftrightarrow A = C$ or $B = C$.
- lineal if $|A : C| = 2$, $|B : C| = 2$.
- gen. type otherwise.

H.W. $BS(1,2) = \langle a, b \mid b^{-1} a b = a^2 \rangle = HNN(\mathbb{Z})$.

What type does $BS(1,2)$ on the B-S tree have?