Eg: $G \curvearrowright$ tree. Then every element is ell. or lox.

More generally, a f.g. gp G cannot act parabolically on a tree.

Eg: $G = A *_C B \curvearrowright$ Bass-Serre tree.

The action is

- elliptic $\Leftrightarrow A=C \overset{\text{or}}{,} B=C$
- lineal if $|A:C|=2$, $|B:C|=2$.
- gen. type otherwise.

H.W: $BS(1,2) = \langle a, b \mid b^{-1}ab = a^2 \rangle = HNN(\mathbb{Z})$.

What type does $BS(1,2)$ on the B-S tree have?

15/5/19

Denis Osin Lecture 2

Correction: If $G \curvearrowright S$ is lineal, then $Fix_G(\partial S)$ has 0 or 2 points.

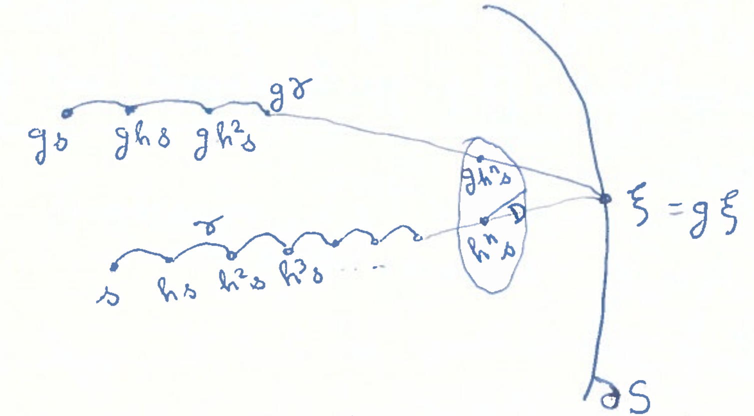
Thm (Gromov): If $G \curvearrowright S$ geometrically (properly, coboundedly) with S hyperbolic, then the action is elliptic, lineal, or of general type.

No parabolic geometric action as orbit under cobounded action is quasiconvex.

Let $G \curvearrowright S_{hyp}$ be quasi-parabolic & geometric.

Thus $\exists\, \xi \in \partial S$ s.t. $g\xi = \xi$ $\forall g \in G$ and $\exists\, h \in G$ s.t. h is loxodromic.

Let $g \in G$.



The path γ is a (λ, C) quasi-geodesic, for some $\lambda = \lambda(h, s, \delta)$, $C = C(h, s, \delta)$.

$\exists\ D = D(\lambda, C) \rightarrow$ "orthogonal" to $\mathrm{stab}_G(h^{\pm\infty})$

$\forall$ finite set $F \subseteq G$, $\exists\ n \in \mathbb{N}$ s.t.

$$\mathrm{Ball}(h^n s, D) \supseteq F h^n s .$$

$\Rightarrow$ The action is <u>not</u> <u>uniformly proper</u>.

An action $G \curvearrowright S$ is <u>uniformly proper</u> if

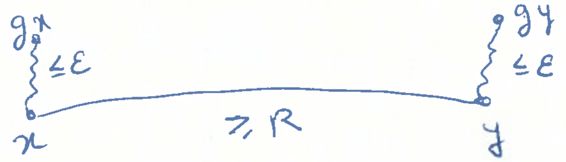
$$\forall r > 0,\ \exists N,\ \forall s \in S,\ |\{g \in G \mid d_S(s, gs) \le r\}| \le N$$

<u>H.W.</u> Geometric $\Rightarrow$ uniformly proper.

<u>Acylindrical action</u>

<u>Def</u>: An action $G \curvearrowright S$ is <u>acylindrical</u> if $\forall \varepsilon > 0$, $\exists\ R, N > 0$ s.t. $\forall x, y \in S$, $d_S(x, y) \geqslant R \Rightarrow$

$$|\{g \in G \mid d_S(x, gx) \le \varepsilon,\ d_S(y, gy) \le \varepsilon\}| \le N$$



Examples

① $\forall$ action on a bounded space is acyl.
(choose $R > \operatorname{diam}(S)$)

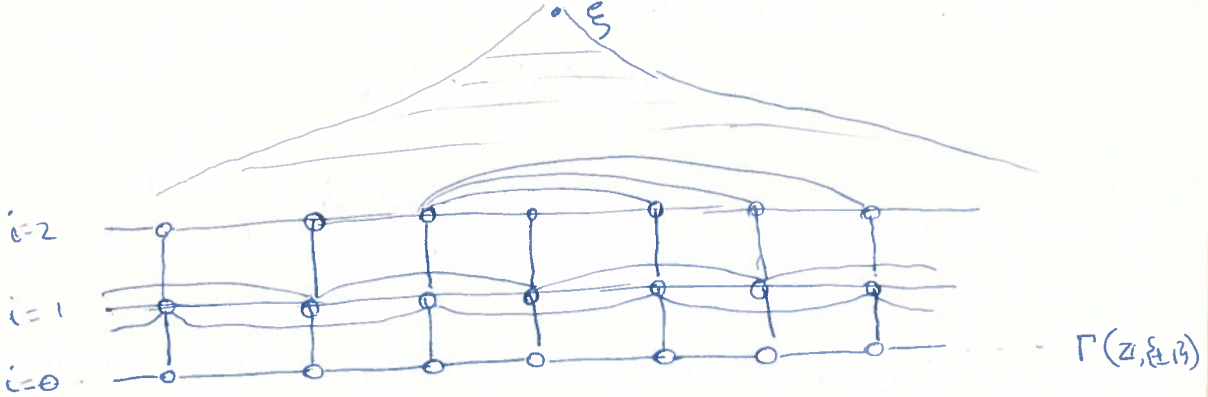
② Geometric $\Rightarrow$ uniformly proper $\Rightarrow$ acylindrical
(In general, proper $\not\Rightarrow$ acylindrical).

Rmk: Every countable group has a proper parabolic action on a proper hyperbolic space.

~~Eg~~ Proof (Groves-Manning)

$G = \mathbb{Z}$. We will construct $\mathcal{H}(\mathbb{Z})$.

For each $i \in \mathbb{N} \cup \{0\}$ take a copy of ~~$\mathbb{Z}$ — F~~ $\Gamma(\mathbb{Z}, \{\pm 1\})$
$\exists$ vertical edges between (m, i), $(m, i+1)$.
At level i, $\exists$ ~~new~~ horizontal edges b/w $(m, i), (m', i)$
$\Leftrightarrow d((m,0),(m',0)) \le 2^i$



$\mathcal{H}(\mathbb{Z})$ is hyperbolic, $\mathbb{Z} \overset{\text{proper}}{\curvearrowright} \mathcal{H}(\mathbb{Z})$,
parabolic as $\partial \mathcal{H}(\mathbb{Z}) = \{\xi\}$

We will see that acylindrical actions are not ~~prof~~ parabolic. $\therefore \mathbb{Z} \curvearrowright \mathcal{H}(\mathbb{Z})$ is not acylindrical.

Let G, $H \leq G$. G is finitely gen. rel to H if $G = \langle X \cup H \rangle$ for some finite X.

Then: $\exists\ \varepsilon : F(X) * H \longrightarrow G$

Let $N = \operatorname{Ker} \varepsilon$. Assume G is f.p. rel to ~~H~~ H.

i.e., $N = \langle\langle R \rangle\rangle^{F(X)*H}$ for some $|R| < \infty$.

Given $w \in \overset{F(X)*H}{\cancel{(X \cup X^{-1})^*}}$ s.t. $\varepsilon(w) = 1$,

$$(*) \quad w =_{F(X)*H} \prod_{i=1}^{k} f_i R_i^{\pm 1} f_i^{-1}, \quad f_i \in F(X)*H,\ R_i \in R$$

$$\boxed{\operatorname{Area}^{rel}(w) = \min k \text{ in } (*)}.$$

Def: G is hyp. rel. to H if G is f.p. rel. to H and satisfies a linear isoperimetric inequality,

i.e., $\exists\ C > 0$ s.t. $\forall\ w \in \operatorname{Ker}\varepsilon$,

~~$\operatorname{Area}^{rel}(w) \leq |C| \cdot |w$~~

$$\operatorname{Area}^{rel}(w) \leq C \cdot |w|_{X \cup H}$$

③ Let G be hyp rel. to H, $G \curvearrowright \Gamma(G, X \cup H)$ is acylindrical.

④ $MCG(S_{g,p}) \curvearrowright$ Curve complex $C_{g,p}$ ($p \to$ no. punctures)

(Masur - Minsky) $3g + p \geq 5 \Rightarrow C_{g,p}$ is hyp.

(Bowditch) $3g + p \geq 5 \Rightarrow MCG(S_{g,p}) \curvearrowright C_{g,p}$ is acyl.

⑤ RAAG (freely indecomposable) $\curvearrowright$ Extension graph.

(Kim - Koberda) Ext. graph is hyp, $\curvearrowright$ is acyl.

Def: A group G is *acylindrically hyperbolic* if it admits a non-elementary acyl. action ($|\Lambda(G)| = \infty$) on a hyperbolic space.

Thm (Osin 2016) Let $G \curvearrowright S$, where S is hyp and the action is acyl. Then exactly one of the following conditions is true:

1) The action is elliptic. ~~$(\Leftrightarrow |G| < \infty)$~~
2) The action is linear $(\Leftrightarrow |G| = \infty$ & G is virt cyclic$)$.
3) The action is of general type.

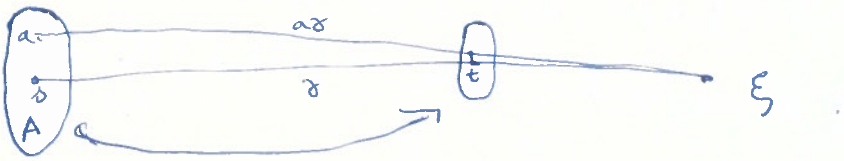
Therefore G is acyl hyp $\Leftrightarrow$ G is not virt. cyclic and has an acyl. action on a hyp. space with unbounded orbits.

Proof of thm: Exclude par. & quasi-par. actions.

① $G \underset{\substack{\text{unif} \\ \text{prop}}}{\curvearrowright} S_{hyp} \Rightarrow$ not par. or q-par.

② $\exists\, r > 0,\ \forall N\ \exists\, s$ s.t. $A = \{g \in G \mid d(s, gs) < r\}$

$$|A| \geq N$$



Thm (Osin, 16) TFAE $\forall G$.

1) G is acyl hyp

2) G has a non-el. cobounded acyl. action on a hyp space.

3) $\exists$ gen. set X of G s.t. $\Gamma(G,X)$ is hyp, $|\partial\Gamma(G,X)| = \infty$, and $G \curvearrowright \Gamma(G,X)$ is acyl.

Thm (Balasubramanya, 18)

G is acyl. hyp $\Leftrightarrow$ G has a non-el. cobounded acyl. action on a quasi-tree

$\Leftrightarrow$ $\exists$ gen. set X of G s.t. $\Gamma(G,X)$ is a quasi-tree, $|\partial\Gamma(G,X)| = \infty$, and $G \curvearrowright \Gamma(G,X)$ is acyl.

Examples: ① Non-elementary hyp. groups

② G is hyp rel. to $H \lneqq G$, G not virt. cyclic $\Rightarrow G$ is a.h.

③. Directly indecomposable RAAG (Kim, Koberda)

④ $MCG(S_{g,p})$, for $(g,p) \neq (0,0), \ldots (0,3)$

⑤ If G is a-h., then $\forall N \lhd G$, $|N| < \infty$ or N is a.h.

Non-examples

1) Groups with infinite amenable normal subgps

2) $G = A \times B$ $|A|, |B| = \infty$.

3) Higher rank lattices ($SL_n(\mathbb{Z}), n \geq 3$)