Non-examples

1) Groups with infinite amenable normal subgps

2) $G = A \times B \quad |A|, |B| = \infty$.

3) Higher rank lattices $(SL_n(\mathbb{Z}), n \geq 3)$

Lecture 3 16/5/2019

Def (Bestvina - Bromberg - Fujiwara): Let $G \curvearrowright S$, $h \in G$. We say that h satisfies the <u>WPD property</u> if $\forall \varepsilon > 0$, $\forall s \in S$, $\exists N \in \mathbb{N}$ s.t.

$$|\{g \in G \mid d(s, gs) \leq \varepsilon \ \& \ d(h^N s, g h^N s) \leq \varepsilon\}| < \infty.$$

<u>Thm</u> (Osin, Bestvina - Bromberg - Fujiwara, Dahmani - Guirardel - Osin).

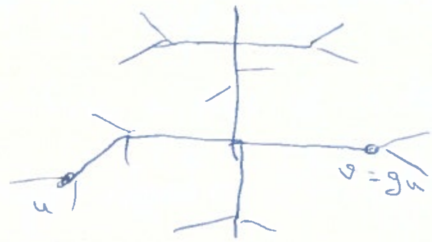
If $G \curvearrowright_{hyp} S$ and $\exists$ a loxodromic WPD element $h \in G$, then G is a.h. or $|G : \langle h \rangle| < \infty$

Example: $\underset{n \geq 2}{Out(F_n)}$ is a.h. (Bestvina - Feighn).

$G = A *_C B$ with C weakly malnormal in ~~A and B~~ G.

We say that $C \leq G$ is weakly ~~non~~ malnormal if $\exists g \in G$ s.t. $|g^{-1} C g \cap C| < \infty$.

$G \curvearrowright T$ Bass-Serre tree



$|\mathrm{stab}_G(\{u, v\})| < \infty$

<u>Lemma (Minasyan, Osin)</u> If $G \curvearrowright T$ minimal and $\mathrm{Fix}_G(\partial T) = \phi$, then $\forall\, u, v \in V(T)$, $\exists$ loxodromic element $h \in G$ s.t. $\mathrm{axis}(h) \supseteq \{u, v\}$

$\therefore$ h is a WPD element (when C weakly malnormal).

<u>Thm (Minasyan - Osin)</u>

a) Let $G = A *_C B$, where $A \neq C \neq B$ and C is weakly malnormal in G. Then G is virt. cyclic or ~~malnormal~~ a.h.

b) Let $G = A *_{C^t = D}$. Assume $C \neq A \neq D$ and C is weakly malnormal in G. Then G is a.h.

Ex: $\mathrm{Aut}(k\langle x, y\rangle)$ free assoc. alg. over a field k. is a.h. $\forall$ field k.

<u>H.W.</u> $H = \langle a, b, c, d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 \rangle$ is a.h.

Thm (Minasyan - Osin) $G = \pi_1(M)$, where M connected, compact, orientable (irreducible) 3-manifold. Then exactly one of the following holds:

1) G is a.h.

2) M is Seifert fibered

3) G is not soluble.

Other examples of a.h. groups

- Groups of deficiency ≥ 2, i.e.,

  $\exists\ G = \langle X \mid R \rangle$, $|X|, |R| < \infty$ and $|X| - |R| \geq 2$,

  then G is a.h. (Osin).

Open problem: Find an "elementary" proof.

- $C'(1/6)$ groups (Gruber - Sisto)
- Convergence groups (Sun)

Properties of a.h. groups

- Every a.h. group has finite amenable radical (maximal normal amenable subgp). (denoted by $K(G)$).
- A.h. $\Rightarrow$ not inner amenable
- A.h. $\Rightarrow$ not boundedly generated (i.e. $G \neq A_1 \cdots A_k$, $A_i = \langle a_i \rangle$)

  More generally, $G = A_1 \cdots A_k$, then $\exists\ i$ s.t. A_i is a.h.

Def G is SQ-universal if $\forall$ countable group embeds in a quotient of G.

Non-example: finite gps, amenable gps.

- F_2 is SQ-universal (HNN-49).
- non-el. hyp $\to$ SQ-universal (Olshanskii)

$SQ \Rightarrow G \geq F_2$.

$\underset{+f.g.}{SQ} \Rightarrow G$ has uncountably many normal subgps

Thm (D-G-O) Every a.h. gp is SQ-universal.

Thm Let Γ be a lattice in a connected Lie group. If Γ is a.h., then it is commensurable to a lattice in a non-compact simple Lie group of rank 1. In particular, Γ is non. el. (rel) hyperbolic.

Proof: Γ a lattice in G, S = solvable radical of G. $\Rightarrow \underbrace{S \cap \Gamma}_{\text{finite} \because \Gamma \text{ is a.h.}}$ is a lattice in S and

$\Gamma / S \cap \Gamma$ is a lattice in G/S - semi simple

$$G/S = M_1 \times M_2 \times \dots \times M_k$$

Wlog, Γ is a lattice in $\underbrace{M_1 \times \dots \times M_k}_{\text{simple}}$.

① $\exists \geq 2$ non-compact factors Not possible.

Monod-Shalom: $H_b^2(\Gamma, \ell^2(\Gamma)) = 0$

Hamenstadt: Γ is a.h. $\Rightarrow H_b^2(\Gamma, \ell^2(\Gamma)) = \infty$

2) $\exists \leq 1$ non-compact factor

Γ is commensurable to a lattice in a simple ~~Lie~~ Lie group M.

Margulis normal subgp thm + SQ universality

$\underbrace{\text{Margulis normal}}$

higher rank lattice has contably many normal subgps

$\Rightarrow$ rank $M = 1$.

Thm (Sisto)

Let $G \underset{\text{acyl}}{\curvearrowright} \underset{\text{hyp}}{S}$ be f.g.. Then the probability that the simple random walk arrives at a loxodromic element after n steps is $1 - O(\varepsilon^n)$, $\varepsilon \in (0,1)$.

Techniques

1) Hyperbolically embedded subgroups (D-G-O).
   - Group theoretic Dehn filling
2) Small cancellation theory (Hull)
3) Monod - Shalom rigidity theory for measure preserving actions.

$\boxed{\text{D-Osin: Groups acting acyl. on hyp spaces, Proc. ICM - 2018}}$

Open Questions: 1) Is acylindrical hyperbolicity a Q.I. invariant of f.g. groups?

1') Let G be a.h., ~~H≥G~~ $G \leq H$, $[H:G] < \infty$. Is H a.h?

1) $\beta_1^{(2)}(G) > 0 \iff G$ is non-amenable & $\exists\, q: G \to \ell^2(G)$ unbdd s.t. $\forall g, h \in G$, $q(gh) = q(g) + \lambda_G(g)\, q(h)$

2) $\beta_1^{(2)}(G) > 0$ & G is f.p. $\overset{?}{\Longrightarrow}$ G is a.h.

Rmk: not true ∞-presented gps (Lück - O)

3) Is acylindrical hyperbolicity a measure equivalence invariant ?