Roman Sauer, Intro to ℓ^2-Betti numbers

Lecture 1

Goal: discussion of Kaplansky's zero division conjecture:

Conj: Γ torsion free group $\Rightarrow$ $\mathbb{C}[\Gamma]$ has no zero divisors i.e. if $a,b\in \mathbb{C}[\Gamma]\setminus\{0\}$ then $a\cdot b\neq 0$.

Also we will investigate what the theory of ℓ^2-Betti numbers brings to the table to attack this conjecture.

Tools:

- functional analysis
- non commutative ring theory
- approximation.

Introduction and Atiyah conjecture

0) Motivation: why Hilbert space methods are relevant to the study of $\mathbb{C}[\Gamma]$?

Linear algebra:

$A,B\in M_{n\times n}(\mathbb{C})$, If $A\cdot B=I_{n\times n}$ then $B\cdot A=I_n$. - easy fact, consequence of vector space dimension.

Direct finiteness conjecture:

Let F be a field. Let $a,b\in F[\Gamma]$. If $a\cdot b=1$ then $b\cdot a=1$.

For finite Γ, $F[\Gamma]$ is finite-dimensional: in that case see above (regard a,b as linear endomorphisms $F[\Gamma]\to F[\Gamma]$)

In general, we need more elaborate tools, for example:

1) von Neumann dimension

analogy: if we have $U\subset \mathbb{C}^n$ subspace and a ~~proj~~ idempotent $p\in M_{n\times n}(\mathbb{C})$: $p^2=p$, $\mathrm{im}\,p=U$ (e.g. projection onto U)

then $\dim_{\mathbb{C}}(U)=\mathrm{tr}_{\mathbb{C}}(p)$

Let us remind

$$\mathbb{C}[\Gamma]=\Big\{\sum_{\gamma\in\Gamma} a_\gamma\cdot\gamma \;\Big|\; a_\gamma\in\mathbb{C},\ a_\gamma=0 \text{ for almost all } \gamma\in\Gamma\Big\}$$

- $\mathbb{C}$-algebra w/ multiplication

$$\Big(\sum a_\gamma\gamma\Big)\Big(\sum b_\gamma\gamma\Big)=\sum_\gamma\Big(\sum_{\gamma_1,\gamma_2} a_{\gamma_1}b_{\gamma_2}\Big)\gamma$$

where $\gamma_1\cdot\gamma_2=\gamma$

dense

$$\ell^2(\Gamma)=\Big\{\sum_{\gamma\in\Gamma} a_\gamma\cdot\gamma \;\Big|\; a_\gamma\in\mathbb{C},\ \sum|a_\gamma|^2<\infty\Big\}$$

- Hilbert space with basis Γ

$$\Big\langle\sum a_\gamma\cdot\gamma,\sum b_\gamma\cdot\gamma\Big\rangle=\sum\bar{a}_\gamma\cdot b_\gamma$$

It comes with the natural isometric Γ-action - exercise (mental exercise..)

$$\Big(\sum a_\gamma\gamma\Big)^*=\sum\bar{a}_\gamma\cdot\gamma^{-1}\qquad (ab)^*=b^*a^*$$

There is an embedding

$$\mathbb{C}[\Gamma]\xhookrightarrow{\ r\ } B(\ell^2[\Gamma])^\Gamma=\{\text{bounded operators on } \ell^2[\Gamma]\}^\Gamma$$

$\leftarrow$ equivariant ~~w~~ w/resp. to left action $\Gamma\curvearrowright\ell^2(\Gamma)$

$$a\longmapsto r_a\qquad r_a(x)=x\cdot a$$

Define $B(\ell^2[\Gamma])^\Gamma=:L(\Gamma)$ group von Neumann algebra

(analogous to $M_{n\times n}(\mathbb{C})$)

Note $r_{ab}=r_b\circ r_a$, $r_{a^*}=(r_a)^*$

Definition A Hilbert $L(\Gamma)$-module is a Hilbert space V endowed with a linear isometric action of Γ s.t. there exists an isometric, Γ-equivariant, linear embedding $V \hookrightarrow \ell^2(\Gamma) \hat{\otimes} \mathcal{H}$ where $\mathcal{H}$ is some Hilbert space.

We say V is finitely generated if $\mathcal{H}$ is finite dimensional; in this case $V \hookrightarrow (\ell^2[\Gamma])^n$ i.e. V is a closed Γ-invariant subspace of $\ell^2(\Gamma)^n$

analogous to $\mathbb{C}$-vector space

Remark V does not come with a choice of $\mathcal{H}$.

Definition The vN trace $tr_{L(\Gamma)}$ on $L(\Gamma)$ is the linear functional

$$tr_{L(\Gamma)}(T) = \langle T(e), e \rangle_{\ell^2(\Gamma)} \in \mathbb{C}$$

Remark $tr_{L(\Gamma)}$ satisfies the trace property i.e.

$$tr_{L(\Gamma)}(S.T) = tr_{L(\Gamma)}(S.T) \quad \text{for } S,T \in L(\Gamma)$$

Example $S = r_a$, $T = r_b$, where $a = \sum a_\gamma \gamma$, $b = \sum b_\gamma \gamma \in \mathbb{C}[\Gamma]$ $\leftarrow$ in general follows from closeness of r + continuity

$$tr_{L(\Gamma)}\Big(r_{\underset{\mathbb{C}[\Gamma]}{\underset{\cap}{\sum c_\gamma \gamma}}}\Big) = \langle \sum \overline{c}_\gamma \gamma, e \rangle_{\ell^2(\Gamma)} = c_e$$

(coefficient at identity)
(unit coefficient)

unit coefficient of ab and ba is $\sum a_\gamma b_{\gamma^{-1}}$.

Remark the vN trace can be extended to endomorphisms of Hilbert $L(\Gamma)$-modules:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook] & V \arrow[d, hook] \\ \ell^2\Gamma \hat{\otimes} \mathcal{H} & \ell^2\Gamma \hat{\otimes} \mathcal{H} \end{tikzcd}$$

f - morphism i.e. bounded and Γ-equiv. Pick a basis $\{h_i\}_{i \in I}$ of $\mathcal{H}$

$$tr_{L(\Gamma)} f = \sum_{i \in I} \langle f(e \otimes h_i), e \otimes h_i \rangle$$

If $\mathcal{H}$ is fin. dim. (so $\cong \mathbb{C}^n$), view $f = f_{ij} \in M_{n \times n}(L(\Gamma))$ and $tr_{L(\Gamma)} f = \sum_{i=1}^{n} tr_{L(\Gamma)}(f_{ii})$

Example
- Γ - finite

$$tr_{L(\Gamma)} f = \sum_{i \in I} \langle f(e \otimes h_i), e \otimes h_i \rangle = \sum_{\gamma \in \Gamma} \sum_{i \in I} \langle \gamma f(e \otimes h_i), \gamma \otimes h_i \rangle = \frac{1}{|\Gamma|} \sum_{\gamma} \sum_{i} \langle f(\gamma \otimes h_i), \gamma \otimes h_i \rangle$$

$$= \frac{1}{|\Gamma|} \cdot tr_{\mathbb{C}} f$$

- $\Gamma = \mathbb{Z} = \langle t \rangle$ $\quad$ $L(\mathbb{Z}) \circlearrowright \ell^2(\mathbb{Z})$ $\quad \xleftrightarrow{\text{Fourier transform}} \quad$ $L^2(S^1) \circlearrowleft L^\infty(S^1)$ $\leftarrow$ action by pointwise multiplication

$tr_{L(\mathbb{Z})}$ $\qquad$ $L^\infty(S^1) \to \mathbb{C}$ $\leftarrow$ $f \mapsto \int_{S^1} f \, d\mu$

$\mathbb{C}[\mathbb{Z}]$ $\qquad$ $\mathbb{C}[z, z^{-1}] \subset L^\infty(S^1)$

$$\begin{tikzcd}
t \arrow[r, mapsto] & z \\
\mathbb{C}[\mathbb{Z}] \arrow[r, "\cong"] \arrow[d, hook] & \mathbb{C}[z, z^{-1}] \arrow[d, hook] \\
L(\mathbb{Z}) \arrow[r, "\cong"] \arrow[dr, "tr_{L(\mathbb{Z})}"'] \arrow[dr, phantom, shift left=3ex, "\#"] & L^\infty(S^1) \arrow[d, "\int_{S^1} \cdot \, d\mu"] \\
& \mathbb{C}
\end{tikzcd}$$

diagram for above correspondence

Definition Let V be a Hilbert $L(\Gamma)$-module. Pick $V \hookrightarrow \ell^2(\Gamma) \hat{\otimes} \mathcal{H}$. Let $pr_V : \ell^2(\Gamma) \hat{\otimes} \mathcal{H} \circlearrowleft$ be the orthogonal projection onto V.

Set

$$\dim_{L(\Gamma)} V = tr_{L(\Gamma)}(pr_V) \in [0, \infty]$$

(analogous to $\dim_{\mathbb{C}}$)

Note

$$\dim_{L(\Gamma)} V = \sum \langle pr_V(e \otimes h_i), e \otimes h_i \rangle \overset{\text{since } pr_V \text{ projection}}{=} \sum \langle pr_V(e \otimes h_i), pr_V(e \otimes h_i) \rangle.$$

So if $\dim_{L(\Gamma)} V = 0$ then $pr_V = 0$ thus $V = 0$ (faithful)

Example $\Gamma = \mathbb{Z}$ $\quad \{ f \in L^\infty(S^1) \mid f_{|A^c} \equiv 0 \} \subseteq L^2(S^1)$ invariant closed subspace, $A \subseteq S^1$ Borel

$$pr_V = \chi_A \qquad \dim_{L^\infty(S^1)}(V) = \mu(A) \in [0,1]$$

Example $q \in L(\Gamma)$ idempotent

$im(q) \subset \ell^2(\Gamma)$ closed Γ-equivariant subspace

statement:

$$\dim_{L(\Gamma)}(im\, q) = tr_{L(\Gamma)}\, q$$

Lemma Let p be orth. projection onto $im\, q$. Then

$$tr_{L(\Gamma)}(p) = tr_{L(\Gamma)}(q)$$

Proof $pq = q$, $qp = p$ – clear

$x = p - q$, then $x^2 = p + q - q - p = 0$.

$s := 1 + x$ is invertible with $s^{-1} = 1 - x$

$sps^{-1} = \ldots = q$

So $tr(p) = tr(sps^{-1}) = tr(q)$ by trace property. #

Proof of direct finiteness for $F = \mathbb{C}$

$ab = 1$ in $L(\Gamma)$ $\Rightarrow$ ba is idempotent since $b(ab)a = ba$.

$$\dim_{L(\Gamma)}(im(1 - ba)) = tr_{L(\Gamma)}(1 - ba) \underset{\text{trace prop.}}{=} 1 - tr_{L(\Gamma)}(ab) = 0$$

faithful $\Rightarrow$ $im(1 - ba) = 0$ $\Rightarrow$ $ba = 1$ #

Using $\dim_{L(\Gamma)}$ one can define ℓ^2-Betti numbers:

2) ℓ^2-Betti numbers

X CW-complexes with cell permuting action of Γ (prime example: $\Gamma = \pi_1 Y \curvearrowright \tilde{Y}$ universal covering)

$$\dots \to C_i X \to C_{i-1} X \to C_{i-2} X \to \dots$$

Assume $\Gamma \curvearrowright X$ is free, finitely many Γ-orbits of i-cells for every $i \in \mathbb{N}$. Then $C_i X \cong \mathbb{Z}[\Gamma]^{n_i}$, $n_i = \#\,\Gamma$-orbits of i-cells
(*)

$$C_i^{(2)}(X_i;\Gamma) := \ell^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_i(X) \cong \ell^2(\Gamma)^{n_i}, \quad \text{differential } d_i^{(2)} = \mathrm{id} \otimes d_i$$

morphisms of Hilbert $L(\Gamma)$-modules

$$\overline{H}_i^{(2)}(X;\Gamma) = \ker d_i^{(2)} \Big/ \overline{\operatorname{im} d_{i+1}^{(2)}} \quad \leftarrow \text{taking closure makes this a Hilbert } L(\Gamma) \text{ module}$$

$$\overline{\operatorname{im} d_{i+1}^{(2)}}^{\perp} \cong \dots \hookrightarrow \ker d_i^{(2)} \hookrightarrow C_i^{(2)}(X;\Gamma) \cong \ell^2(\Gamma)^{n_i}$$

Define $b_i^{(2)}(X;\Gamma) := \dim_{L(\Gamma)} \overline{H}_i^{(2)}(X;\Gamma) \in [0, n_i]$ ℓ^2-Betti number of X.

3) Atiyah conjecture

Conjecture (for torsion free Γ) AC

① The ℓ^2-Betti numbers of a free Γ-CW-complex of finite type (*) are integers provided Γ is torsion free.

Conjecture (torsion free AC for matrices over F) ← field

② Let $A \in M_{m\times n}(F[\Gamma])$, $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. If Γ is torsion free then $\dim_{L(\Gamma)}\left(\ker\left(r_A : \ell^2(\Gamma)^m \to \ell^2(\Gamma)^n\right)\right) \in \mathbb{Z}$.

Conjecture Zero divisor conjecture over F

③

Relations ① ⟹ ② for $F = \mathbb{Q}$: realise A as $C_3(X) \to C_2(X)$ in a suitable free Γ-CW-cplx of dim 3.

② ⟹ ① for any F

② ⟹ ③

$a, b \in F[\Gamma] \setminus \{0\}$ with $ab = 0$ $\Rightarrow \ker r_b \neq \{0\}$ ($\subseteq \ell^2(\Gamma)$). But $\dim \ker r_b \in \mathbb{Z}$ so $\dim \ker r_b = 1$.

That means $\ker r_b = \ell^2(\Gamma) \Rightarrow b = 0$ ↯

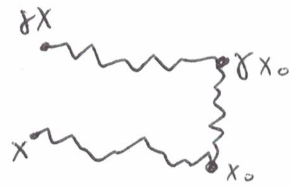
Proof of ② for $\Gamma = \mathbb{F}_2$, $F = \mathbb{C}$. (P. Linnell)

Let $T = (V, E)$ be the standard Cayley graph for $\mathbb{F}_2$. Let $x_0 \in V$ be a base point.

$$V \xrightarrow{f} E \cup \{*\} \quad \text{bijection}$$

$$x \longmapsto \begin{cases} \text{first edge on geodesic from } x \text{ to } x_0 \\ * \quad \text{if } x = x_0 \end{cases}$$

f is almost equivariant i.e. for every $\gamma \in \mathbb{F}_2$ there are finitely many $x \in V$ with $f(\gamma x) \neq \gamma f(x)$



$L(\mathbb{F}_2)$-representations

$$\ell^2(V) \cong \ell^2(\mathbb{F}_2)$$
$$\ell^2(E) \cong \ell^2(\mathbb{F}_2) \oplus \ell^2(\mathbb{F}_2)$$

$\rho_+ : L(\mathbb{F}_2) \to B(\ell^2(V))$ regular rep.

$\rho : L(\mathbb{F}_2) \to B(\ell^2(E))$

$\rho_- : L(\mathbb{F}_2) \to B(\ell^2(V))$ $\quad \rho_-(a) = F^{-1} \circ \rho(a) \circ F$ where F is induced by f:
$F : \ell^2 V \to \ell^2 E \oplus \mathbb{C}$

Lemma 1 $\rho_+(a) - \rho_-(a)$ is of finite rank for $a \in \mathbb{C}[\mathbb{F}_2] \subset L(\mathbb{F}_2)$

Proof follows from almost-equivariance

Lemma 2 For all $a \in \mathbb{C}[\mathbb{F}_2]$ (or $a \in L(\Gamma)$ of finite rank - fin. dim. image in the sense of $\mathbb{C}$-dimension)

Claim there exist non-trivial ones? See remark below!

$$\operatorname{tr}_{L(\mathbb{F}_2)}(a) = \sum_{x \in V} \langle (\rho_+(a) - \rho_-(a))(x), x \rangle$$

Proof computation

Lemma 3 $p, q \in B(\mathcal{H})$ projections s.t. $p - q$ has finite rank then

$$\operatorname{tr}_{\mathbb{C}}(p-q) \in \mathbb{Z}$$

Proof
$(p-q)^2$ is self-adjoint, finite rank (p, q commute with $(p-q)^2$ but not with $(p-q)$)

$$\mathcal{H} = \ker(p-q)^2 \oplus \bigoplus_\lambda E_\lambda \leftarrow \text{eigenspaces}$$
$$\operatorname{tr}_{\mathbb{C}}(p-q) = \sum_\lambda \left(\operatorname{tr}_{\mathbb{C}} p_{|E_\lambda} - \operatorname{tr}_{\mathbb{C}} q_{|E_\lambda} \right) \in \mathbb{Z}.$$

Lemma 4 $a, b \in B(\mathcal{H})$ with $a - b$ finite rank $\Rightarrow pr_{\ker(a)} - pr_{\ker(b)}$ finite rank

Proof excercise

Remark Actually in Lemma 2 we want $a \in L(\Gamma)$ s.t. $\rho_+(a) - \rho_-(a)$ is of finite rank

Proof of AC.

Let $a \in \mathbb{C}[\mathbb{F}_2]$

$$\dim_{L(\mathbb{F}_2)}(\ker r_a) = \operatorname{tr}_{L(\mathbb{F}_2)}(pr_{\ker r_a}) \overset{\text{Lemma 1,2,4}}{=} \sum_{x \in V} \langle (\rho_+(pr_{\ker r_a}) - \rho_-(pr_{\ker r_a}))x, x \rangle = \operatorname{tr}_{\mathbb{C}}\left(\rho_+(pr_{\ker r_a}) - \rho_-(pr_{\ker r_a})\right) \in \mathbb{Z}$$

Lemma 3