FINITE TIME DYNAMICS and PREDICTIONS for CHAOTIC and RANDOM SYSTEMS

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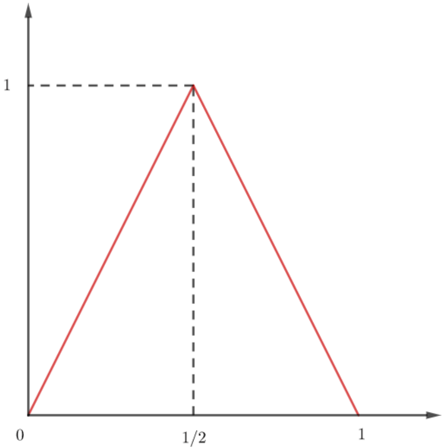
Outline

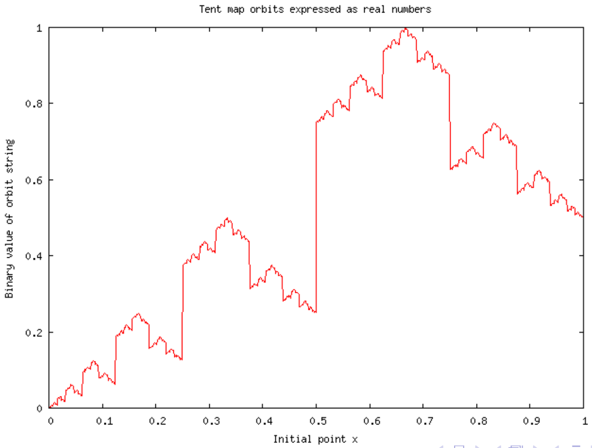
The Tent Map

Consider the tent map defined by $T(x) = 2\min\{x, 1-x\}$ acting on the space $[0,1]$. Pictured is a graph of $T(x)$. If μ is the usual length measure, note that $\mu(T^{-1}I) = \mu(I)$ for any interval I.

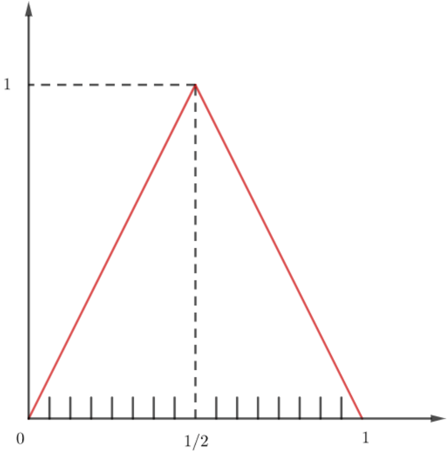


The Tent Map

We can record the time evolution of a point x with a binary sequence $\tilde{x}_0\tilde{x}_1\tilde{x}_2\ldots$ where $\tilde{x}_i = \begin{cases} 0 & \text{if } T^i x \in [0, 1/2) \\ 1 & \text{if } T^i x \in [1/2, 1]. \end{cases}$ Pictured is a graph of $\tilde{x}$.



In practice, we want to make predictions about where points will go over time. The chaotic nature of the map makes precise predictions difficult. Instead of focusing on individual points, standard approach is break up a phase space into disjoined subsets, i.e consider a partition (coarse graining).



The Tent Map

In general, a point x belongs to an interval A if and only if the sequence representing x begins with some fixed sequence $w_k \dots w_1$, where k and the characters w_i are determined by the specific choice of A. We refer to $w_k \dots w_1$ as the *symbolic representation of A*.

Define $P_A(n)$ to be the probability that some point x first hits the interval A at time n (this is the 'first hitting' probability).

A point x enters A after n iterates of the map T if and only if the following two conditions are met:

- $\tilde{x}_{n-k+1} \dots \tilde{x}_n = w_k \dots w_1$
- $\tilde{x}_{n-i+1} \dots \tilde{x}_{n-i+k} \neq w_k \dots w_1$ for every $k < i \leq n+1$

where $w_k \dots w_1$ is the representation of A.

Calculating the probability that a point enters A after n steps reduces to counting the number of sequences of finite length n that meet the previous conditions.

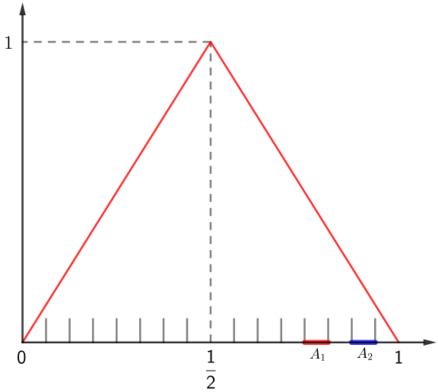
In order to do this, we consider the *autocorrelation* of a word $w = w_k \dots w_1$, denoted $\text{cor}(w)$. It is defined like this: $\text{cor}(w) = b_k \dots b_1$ where $b_i = 1$ if $w_k \dots w_{k-i+1} = w_i \dots w_1$ and $b_i = 0$ otherwise. In other words, $b_i = 1$ if the first i elements of w equal the last i elements.

For example, cor(every eve) $= 100000101$.

Autocorrelations of words(patterns) always start with 1.

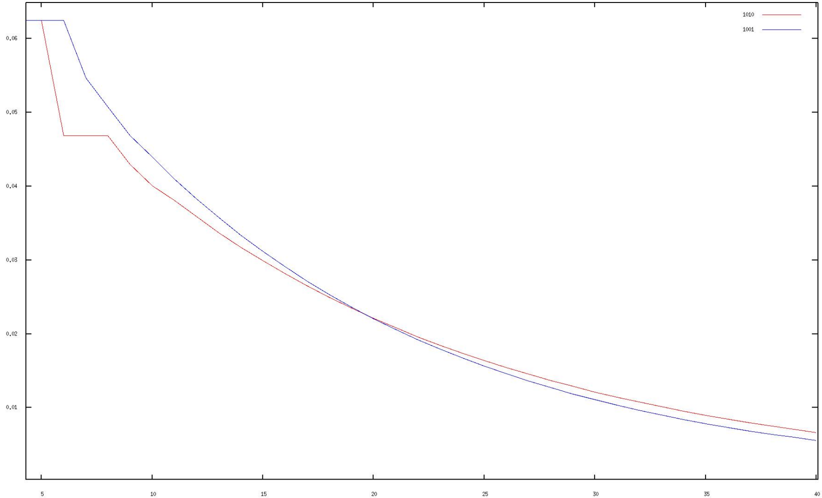
The Tent Map

Let's pick two intervals A_1 and A_2 and compare $P_{A_1}(n)$ and $P_{A_2}(n)$. Observe that the representation of A_1 is 1010 and the representation of A_2 is 1001. Also, note that $\text{cor}(1010) = 10 > \text{cor}(1001) = 9$.



The Tent Map

Pictured are graphs of $P_{A_1}(n)$ and $P_{A_2}(n)$. There exists only one moment in time when the ordering of the first hitting probabilities switches.



The Tent Map

Let w be the representation of A_1 and w' be the representation of A_2, where A_1 and A_2 are any subsets of the domain of a map f, as described above.

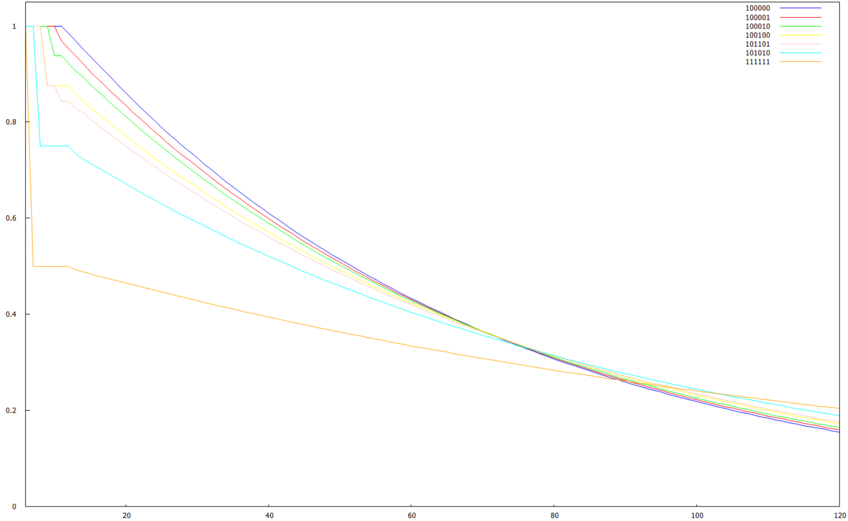
Theorem

Suppose that $cor(w) > cor(w')$. There exists an N, where $k < N < \infty$, such that $P_{A_1}(n) - P_{A_2}(n) \leq 0$ for $n < N$ and $P_{A_1}(n) - P_{A_2}(n) > 0$ for $n \geq N$.

Moreover, if $\text{cor}(w) = \text{cor}(w')$ then $P_{A_1}(n) = P_{A_2}(n)$ for all n.

The Tent Map

In fact, the ordering of all the first hitting probabilities reverses over time.



Numerical results

Length k	Beginning of Intermediate interval	End of interval
4	20	26
5	37	52
6	70	103
7	135	208
8	264	415

It is easy to see that both lengths of the short times interval and of intermediate intervals exponentially (with the base of exponent equal 2) grow to infinity as the length of a word (k) increases.
Therefore IN THE INFINITE TIME LIMIT ONLY THE SHORT TIMES INTERVAL REMAINS where dynamics is totally different from the one in long times interval. Classical/standard approach deals only with question on what is happening AT THE LIMIT of long times interval.

Conway formula

Coin Tossing
Which pattern will appear the first?
Player A bet is (HHTH...H)=A
Player B bet is (THHH...T)=B
Both patterns/words are of length k
ODDS that Player B wins:
$(cor(A) - cor(A,B))/(cor(B) - cor(B,A))$
THIS FORMULA GIVES ANSWER for AVERAGE over INFINITE TIME INTERVAL
cor(A,B):=correlation of A and B

Correlation of Words

Correlation of words (patterns) A and B is a binary word with the length equal the length of the word A.
Example: A is TOMATO, B is TORUS
cor(A,B)=(000010) cor(B,A)=(00000)

TOMATO	
TORUS	0
TORUS	0
TORUS	0
TORUS	0
TORUS	1
TORUS	0

All correlations of different words start with 0.
Generally cor(A,B) does not equal cor(B,A).

Let $T : M \to M$ be a uniformly hyperbolic dynamical system preserving Borel probability measure μ.

Definition

A uniformly hyperbolic dynamical system preserving Borel probability measure μ is called fair dice like or FDL if there exists a finite Markov partition ξ of its phase space M such that for any integers m and j_i, $1 \leq j_i \leq q$ one has $\mu\left(C_\xi^{(j_0)} \cap T^{-1}C_\xi^{(j_1)} \cap \cdots \cap T^{-m+1}C_\xi^{(j_{m-1})}\right) = \frac{1}{q^m}$ where q is the number of elements in the partition ξ and $C_\xi^{(j)}$ is element number j of ξ.

Statement of Results

Examples include the Tent Map, already discussed, as well as the following.

Example

Let $Tx = qx \pmod 1$ where $x \in M = [0, 1]$ and $q \geq 2$ is an integer, with μ the Lebesgue measure. The corresponding Markov partition is the one into equal intervals $[\frac{i}{q}, \frac{i+1}{q}]$, $i = 0, 1, \ldots, q-1$.

Example

Let $T : z \to z^2$ be defined on the Riemann sphere. Its Julia set $\mathcal{J}$ is the unit circle in the complex sphere. Lyubich's measure μ that equidistributes periodic points in the Julia set is a continuous probability measure invariant with respect to μ. By dividing $\mathcal{J}$ into 2^n intervals of equal measure we get an FDL system.

Statement of Results

Let ξ and ξ' be two Markov partitions of the phase space for an FDL system such that ξ' is a refinement of ξ or vice versa. Given $C_{\xi}^{(j)} \in \xi$ and $C_{\xi'}^{(i)} \in \xi'$, we denote by $P_i(n)$ the probability that x first enters $C_{\xi}^{(i)}$ at time n, and denote by $P_j(n)$ the analogous probability for $C_{\xi'}^{(j)}$.

Theorem

There exists $N < \infty$ such that precisely one of the following statements holds:

- $P_i(n) - P_j(n) \leq 0$ *for* $n < N$ *and* $P_i(n) - P_j(n) > 0$ *for* $n > N$
- $P_i(n) - P_j(n) \geq 0$ *for* $n < N$ *and* $P_i(n) - P_j(n) < 0$ *for* $n > N$
- $P_i(n) = P_j(n)$ *for all* n.

Statement of Results

If $C_\xi^{(i)}$ and $C_\xi^{(j)}$ are any two elements of the same partition having 2^k elements, then

Theorem

$N \geq 4k.$

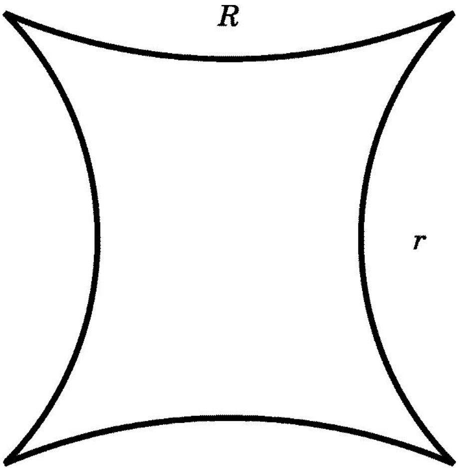
Let $N_{\min}$ be the smallest N among any pair of subsets, and $N_{\max}$ the largest. For $n < N_{\min}$ there is a hierarchy among the first hitting probabilities, and for $n > N_{\max}$ there is an opposite hierarchy.
For $N_{\min} < n < N_{\max}$ the probabilities curves are intersecting, and there is no hierarchy.

Extension of Results

Actually, as numerical experiments have shown, this is a more general phenomenon.

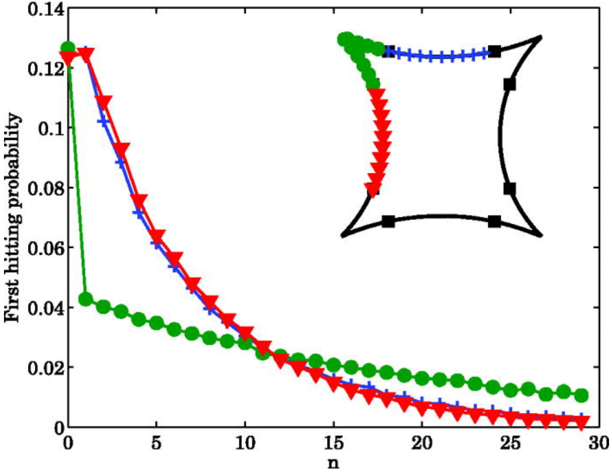
Some Other Examples

The first hitting probabilities for certain hyperbolic billiards behave similarly. Consider the diamond billiard, whose sides are arcs of circles of different radii, acted upon by the return map $\mathcal{F}$ to the boundary:



Some Other Examples

Displayed are the first hitting probabilities for various subsets of the boundary.



Connection to previous Theory

Correlation of words were introduced by John Conway. Fundamental results on their properties were obtained by Guibas and Odlyzko [10].

Bunimovich and Yurchenko proved that the curves of survival probabilities for Markov partitions of Fair Dice Like systems split very fast and then this hierarchy never changes [5]. It was the first (absolutely unexpected) indication that finite time predictions of chaotic dynamics could be possible.
Bolding and Bunimovich [4] proved that indeed finite time predictions can be made. Namely the probabilities of first appearances (visits to) subsets of phase space demonstrate different hierarchies in different time intervals. THERE ARE ONLY THREE INTERVALS. Survival probabilities are integrals /sums of first hitting probabilities, and therefore formally results on FHP are much stronger than previous on SP

We introduce a new map $f_i : M \to M$ as follows

$$f_i(x) = \begin{cases} f(x), & x \notin R_i \\ x, & x \in R_i \end{cases} \quad (1)$$

i.e. each point inside R_i is a fixed point of f_i. Thus we obtain m open dynamical systems generated by $f_i = M \to M, i = 0, 1, \dots, m-1$, induced by f. The partition $\mathcal{R}$ will remain Markov for f_i but the corresponding graph, say G_i, and the transition matrix $A_i = \{a_{sr}(i)\}_{s,r=0}^{m-1}$ become different.

In fact, its entry is $a_{ij} = \delta_{ij}$. In other words, the state "i" is the sink.

Open dynamical system with a given (generating Markov) partition and a chosen its element as a HOLE induces topological Markov chain $tM_C(\sigma, \Omega_{A_i})$. Given $n \geq 0$ let

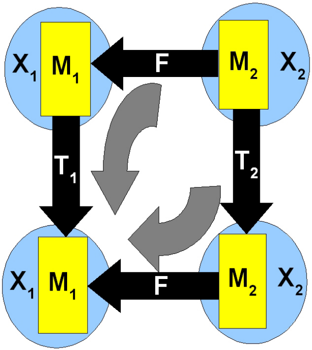
$$\begin{aligned} X(i,n) &= \{x \in M | f^n x \in R_i, f^k x \notin R_i, 0 \leq k < n\} \\ &= \{x \in M | f_i^n \in R_i, f_i^k x \notin R_i, 0 \leq k < n\} \\ Y(i,n) &= \{x \in M | f^k x \in R_i, \text{for some } k, 0 \leq k \leq n\} \\ &= \{x \in M | f_i^k x \in R_i, \text{for some } k, 0 \leq k \leq n\} \end{aligned}$$

So $X(i,n)(Y(i,n))$ consists of the points in M such that the trajectory of the DS generated by f going through each of them intersects R_i for the first time at the instant n (at the instant k, $0 \leq k \leq n$).
It is clear that, $X(i,n') \cap X(i,n'') = \emptyset, n' \neq n''$, and $Y(i,n) = \cup_{l=0}^{n} X(i,l), Y(i,n) = \{x \in M, f^n(x) \in R_i\}$.

Therefore, for an arbitrary measure on M

$$\mu(Y(i,n)) = \sum_{l=0}^{n} \mu(X(i,l)).$$

The measure $\mu(X(i,n))$ can be treated as the probability that an orbit of f hits the hole R_i for the first time at the instant n, and $\mu(Y(i,n))$ is the probability that an orbit hit the hole R_i at some instant k, $0 \leq k \leq n$. Moreover, the quantity $P_n(f_i) = 1 - \mu(Y(i,n)) = \sum_{l=m+1}^{\infty} \mu(X(i,l))$. is a survival probability at the instant of time n.



Definition

Let T_i be a measure-preserving transformation of the Lebesgue probability space $(X_i, \mathcal{B}_i, \lambda_i)$, $i = 1, 2$. We say that T_1 and T_2 are metrically conjugate if there exist $M_i \in \mathcal{B}_i$ with $\lambda_i(M_i) = 1$ and $T_i(M_i) \subset M_i$ and there is a invertible measure-preserving transformation $F : M_2 \to M_1$ such that $\forall x \in M_2$,

$$F \circ T_2(x) = T_1 \circ F(x).$$

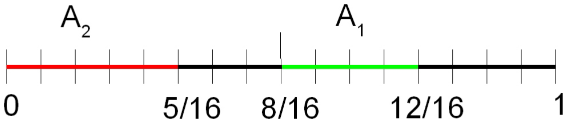
Metric conjugacy and escape rate

Lemma

Let T_1 *and* T_2 *be two metrically conjugate measure-preserving transformations on the Borel probability spaces* $(X_1, \mathcal{B}_1, \lambda_1)$ *and* $(X_2, \mathcal{B}_2, \lambda_2)$, *correspondingly, with a conjugacy map* $F : (\mathcal{B}_2, \lambda_2) \to (\mathcal{B}_1, \lambda_1)$. *Suppose also that for some* $A \in \mathcal{B}_2$ *escape rate into* A *exists.*

$$\rho_{T_2}(A) = \rho_{T_1}(F(A)), \quad \tau_{T_2}(A) = \tau_{T_1}(F(A)).$$

Smaller escape through a larger hole



Q: $\lambda(A_1) < \lambda(A_2)$, but $\rho(A_1) > \rho(A_2)$?

- Lemma: if $B \subset \hat{T}^{-k}A$ for some $k > 0$, then $\rho(A \cup B) = \rho(A)$
- Let $A = [0, \frac{1}{4}]$, $B = [\frac{1}{4}, \frac{5}{16}]$, and $A_1 = [\frac{1}{2}, \frac{3}{4}]$
- Consider two holes: $A_2 = A \cup B$ and A_1
- Easy to check: $B \subset \hat{T}^{-2}A$
- Lemma $\Rightarrow \rho(A) = \rho(A \cup B) = \rho(A_2)$
- Easy to check: $\tau(A) = 1$ and $\tau(A_1) = 3$
- Main Theorem $\Rightarrow \rho(A) < \rho(A_1)$

$$\rho(A_2) < \rho(A_1), \quad \lambda(A_2) > \lambda(A_1)$$

Smaller escape through a larger hole

Dual statement to Poincare theorem on recurrences.

Theorem

For any $\varepsilon \in (0,1)$ *and any* $r > 0$ *there exists a measurable set* $A \subset [0,1]$ *such that*

$$\lambda(A) > 1 - \varepsilon, \quad \rho(A) < r.$$

- There are holes of measure arbitrarily close to one with arbitrarily small escape rate
- In the proof we construct this set - a finite union of intervals

Some examples of FDL-systems

- Linear expanding maps:

$$\hat{T}(x) = \kappa x \mod 1, \quad \kappa \in \mathbb{N}, \kappa > 1$$

- Tent map:

$$\hat{T}(x) = \begin{cases} 2x \mod 1, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x \mod 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

- Logistic map (non-linear map):

$$\hat{T}(x) = 4x(1 - x), \quad x \in [0, 1]$$

- Baker's map (two dimensional invertible map):

$$\hat{T}(x, y) = \left(2x \mod 1, \frac{1}{2}(y + \lfloor 2x \rfloor) \mod 1\right)$$

Summary

1.Finite Time predictions of important characteristics of evolution of random and chaotic systems are possible.
2.It is shown that such predictions can be based on a rigorous mathematical theory.
3.Currently was developed such theory for the very strongly random and very strongly chaotic systems. These results are new not just for dynamical systems theory but for probability theory as well.
4.Although it may seem contradictory that evolution of more random/chaotic systems is easier to predict, but experience with theory of chaotic dynamical systems and with probability theory shows that it is in fact not surprising.
5.Developed theory demonstrates that in systems in UNIQUE equilibrium (no phase transitions) transport in phase space can be very inhomogeneous.

Summary

6.Developed theory offers a new view on a role of periodic orbits in strongly chaotic and random systems. (Particularly it shows that finite time predictions are possible and simpler for systems where finite time Lyapunov exponents is UNIFORM at any moment of time).
7. One (of many) applications is that in transitive networks there are hierarchies of elements with respect to a speed of absorbtion and transmission of "information"(any substance that travels over network).
8. Behavior in short times interval is, in some respects, opposite to what happens for large times. Therefore it makes sense to take it into account and in particular reconsider meanings of various accepted characteristics of evolution/dynamics which involve infinite time limit.
"IN A LONG RUN WE ARE ALL DEAD" Jonh Maynard Keynes

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