Local Time For Ergodic Sums [1]

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Some basic (classic) problems in stochasticity of DS

- Central limit theorem (CLT, IP,...) **Gordin 1968...**
- Rare events (Poisson law, extremal distribition, entrance time...) **Pitskel 1991...**
- Large deviation **Orey 1986...**
- Occupation times **Aaronson 1981...**
- Local limts **Rousseau-Egele 1983...**
- Differentiable statistical funtionals (**Denker, Keller 1983...)**

The Problem

Let $(X, \mathcal{F}, T, \mu)$ be an ergodic p.p. DS, f measurable and $S_n = \sum_{i=0}^{n1} f \circ T^i$ its ergodic sum.

The **occupation time** of $\{S_n\}$ at level t at time n is $\ell(n,t) := \#\{i = 1, 2, \cdots, n : S_i = t\}$; $\ell_n = \ell_n(0)$. Its distribution is called the **local time**.

We are interested in the convergence of these local times and will connect such convergence to local limits. We restrict to functions f with values in $\mathbb{Z}$.

Some literature

Chung and Hunt (1949) studied the limit behavior of the sequence of $\{\ell_n\}$ for simple random walk.
Révész (1981) proved an almost sure invariance principle by Skorokhod embedding.
Borodin (1984): Weak convergence of $\ell(x\sqrt{n}, [nt])/\sqrt{n}$ of a recurrent random walk to the Brownian local time.
Aleškevičienė (1986): Asymptotic distribution and moments of local times of an aperiodic recurrent random walk.
Bromberg and Kosloff (2012): Weak invariance principle for the local times of partial sums of Markov chains.
Aaronson (1981) Distributional convergence in infinite m.p. DS.
Bromberg (2014): Extension to Gibbs-Markov processes.

Conditional Local Limit Theorem

Definition

A centered integer-valued function f is said to have a **conditional local limit theorem at** 0, if there exists a constant $g(0) > 0$ and a sequence $\{B_n\}$ of positive real numbers, such that for all $x \in \mathbb{Z}$

$$\lim_{n\to\infty} B_n\mu(S_n = x|(f \circ T^{n+1}, f \circ T^{n+2}, \ldots) = \cdot) = g(0) \tag{1}$$

almost surely.

The full formulation of the corresponding form of a local limit theorem goes back to Shepp and Stone in the 60es and reads in the conditional form as

$$\lim_{n\to\infty} B_n\mu(S_n = k_n|(f\circ T^{n+1}, f\circ T^{n+2}, \dots) = \cdot) = g(\kappa) \tag{2}$$

μ a.s. as $\frac{k_n - A_n}{B_n} \to \kappa$, for all $\kappa \in \mathbb{R}$, where A_n is some centering constant, and g is the densitiy of some distribution.
Condition (1) can be reformulated using the transfer operator P_T.
The local limit theorem at 0 then reads

$$\lim_{n\to\infty} B_n P_T^n(\mathbb{I}_{\{S_n=t\}}) = g(0) \qquad \text{for all } t\in\mathbb{Z},\ \mu\text{-a.s..} \tag{3}$$

Weak Convergence Theorem

Theorem (Convergence of local times)

Let the $\mathbb{Z}$-valued function f have a conditional local limit theorem at 0 as in (1) with regularly varying scaling constants $B_n = n^\beta L(n)$, where $\beta \in [\frac{1}{2}, 1)$ and L is a slowly varying function.

Put $a_n := g(0) \sum_{k=1}^{n} \dfrac{1}{B_k} \to \infty$.

Then $\dfrac{\ell_n}{a_n}$ converges to a r.v. Y_α strongly in distribution, i.e.

$$\int_X g\left(\frac{\ell_n(x)}{a_n}\right) H(x) d\mu(x) \to E[g(Y_\alpha)], \tag{4}$$

for any bounded and continuous function g and any probability density function H on $(X, \mathcal{F}, \mu)$, and Y_α has the normalized Mittag-Leffler distribution of order $\alpha = 1 - \beta$.

Remark. In this theorem, $\beta \in [\frac{1}{2}, 1)$. This is because $\beta = \frac{1}{d}$, when the local limit holds towards a stable distribution with stability parameter $d \in (0, 2]$. Also, to ensure a_n is divergent, β has to be less than 1.

Strong convergence in distribution is stronger than weak convergence.

A rondom variable Y has **normalized Mittag-Leffler distribution of order** α if

$$E\left(e^{tY}\right) = \sum_{p=0}^{\infty} \frac{\Gamma(1+\alpha)^p t^p}{\Gamma(1+\alpha p)}.$$

For the definitions of strong convergence in distribution and Mittag-Leffler distribution, see J. Aaronson's book 1997 on infinite ergodic theory, Sections 3.6 and 3.7.

Theorem (Deviation and upper bound)

Suppose f satisfies all conditions in Weak Convergence Theorem. Then for every $\gamma > 1$, there exists a constant n_γ such that for all $n_\gamma \le t \le L_2(n)^2$, where $L_2(n) = \log\log n$, one has

$$e^{-\gamma(1-\alpha)t} \le \mu(\ell_n \ge \frac{\Gamma(1+\alpha)}{\alpha^\alpha} t a_{n/t}) \le e^{-\frac{1}{\gamma}(1-\alpha)t}. \tag{5}$$

In addition, if the return time process R_n of ℓ_n is uniformly or strongly mixing from below, where R_n is the waiting time for ℓ_n to arrive at 0 the n-th time, then

$$\limsup_{n\to\infty} \frac{\ell_n}{a_{\frac{n}{L_2(n)}} L_2(n)} = K_\alpha, \text{ a.s.} \tag{6}$$

where $K_\alpha = \dfrac{\Gamma(1+\alpha)}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$.

Corollary (Gibbs-Markov transformation)

Let $(X, \mathcal{F}, T, \mu, \alpha)$ be a mixing, probability preserving Gibbs-Markov map (Aaronson and Denker 2001), and let $\phi : X \to \mathbb{Z}$ be Lipschitz continuous on each $a \in \alpha$, with

$$D_\alpha \phi := sup_{a\in\alpha} D_a \phi = sup_{a\in\alpha} \sup_{x,y\in a} \frac{|\phi(x) - \phi(y)|}{d(x,y)} < \infty$$

and distribution G in the domain of attraction of a stable law with order (stability parameter) $1 < d \leq 2$. Then ϕ has a conditional local limit theorem with $B_n = n^{1/d} L(n)$, where $L(n)$ is a slowly varying function. Moreover, the scaled local time of S_n converges to Mittag-Leffler distribution strongly and (5) holds.
If in addition, the return time process R_n of the local time ℓ_n is uniformly or strongly mixing from below, then (6) holds.

Example

Let $(\Omega, \mathcal{B}, m, S, \alpha)$ is the continued fraction transformation where $\Omega = [0,1]$. It is a mixing and measure preserving Gibbs-Markov map with respect to the Gauss measure $dm = \frac{1}{\ln 2}\frac{1}{1+x}dx$ and the natural partition α given by inverse branches. Define the metric on Ω to be $d(x,y) = r^{\inf\{n: a_n(x) \neq a_n(y)\}}$, where $r \in (0,1)$. Let ϕ be Lipschitz continuous on each set of the partition α, and in the domain of attraction of a stable law with stability parameter $d \in (1,2]$.

Example

...continued.

Define $(X, \mathcal{F}, \mu, T, \beta)$ to be the direct product of $(\Omega, \mathcal{B}, m, S, \alpha)$ with itself and $d_X((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}$. Then one can check that $(X, \mathcal{F}, \mu, T, \beta)$ is still a mixing and measure preserving Gibbs-Markov map. Let $f : X \to \mathbb{Z}$ be defined by $f(x, y) = \phi(x) - \phi(y)$. Since ϕ is Lipschitz on partitions α, so is f. f is in the domain of attraction of a stable law. The local time at level 0 of S_n is denoted to be $\ell_n(x, y) = \sum_{i=1}^n \mathbb{I}_{\{S_i(x,y)=0\}}$. By applying the Corollary to the Gibbs-Markov map $(X, \mathcal{F}, \mu, T, \beta)$ and the Lipschitz continuous function f, S_n has a conditional local limit theorem and the local time ℓ_n converges to the Mittag-Leffler distribution after scaling and (5) holds for ℓ_n. In particular this applies to the number of times that the partial sum $\sum_{j\le i} \phi \circ S^j$ agree at times $i \le n$ when the initial values are chosen independently.

Example (β transformation)

Fix $\beta > 1$ and $T : [0,1] \to [0,1]$ is defined by $Tx := \beta x \mod 1$. Let $\phi : [0,1] \to \mathbb{Z}$ be defined as $\phi(x) = [\beta x]$ and $X_n(x) = \phi \circ T^{n-1}(x) = [\beta T^{n-1} x]$. There exists an absolutely continuous invariant probability measure μ. By Aaronson, Denker, Sarig, Zweimüller 2004, there is a conditional local limit theorem for the partial sum S_n of f. Then the results can be applied to $([0,1], \mathcal{F}, \mu, T)$ and f, it follows that the scaled local time of $S_n - n \int f d\mu$ converges to the Mittag-Leffler distribution and (5) holds when $\int f d\mu$ is an integer. When $\int f d\mu$ is not an integer, a similar product space as in the previous example can be constructed and the same conclusion for the local time in the product space holds.

Sketch of proof

Consider the $\mathbb{Z}$-extension of $(X, \mathcal{F}, T, \mu)$. Define $\tilde{T} : B := X \times \mathbb{Z} \to X \times \mathbb{Z}$ by $\tilde{T}(\omega, n) = (T(x), n + f(x))$, then by induction, $\tilde{T}^k(x, n) = (T^k(x), n + S_k)$. Let ν be the counting measure on the space $\mathbb{Z}$, and $\mathcal{Z}$ be the Borel-σ algebra of $\mathbb{Z}$. So we consider $(B, \mathcal{B}, m = \mu \times \nu, \tilde{T})$. We denote by P_T and $P_{\tilde{T}}$ the transfer operators of T and $\tilde{T}$, respectively.

Lemma

Suppose f has the conditional local limit theorem at 0 (cf. (1)). Then

1. $\tilde{T}$ *is conservative and measure preserving in* $(B, \mathcal{B}, m)$.
2. *There exists a probability space* $(Y, \mathcal{C}, \lambda)$, *and a collection of measures* $\{m_y : y \in Y\}$ *on* $(B, \mathcal{B})$ *such that*
   1. *For* $y \in Y$, $\tilde{T}$ *is a conservative ergodic measure-preserving transformation of* $(B, \mathcal{B}, m_y)$.
   2. *For* $A \in \mathcal{B}$, *the map* $y \to m_y(A)$ *is measurable and*

$$m(A) = \int_Y \mu_y(A) d\lambda(y).$$

3. λ*-almost surely for* y, $(B, \mathcal{B}, \mu_y, \tilde{T})$ *is pointwise dual ergodic.*

From the lemma, $(B, \mathcal{B}, m_y, \tilde{T})$ is pointwise dual-ergodic. Since a_n is regularly varying, by applying Aaronsons version of the Darling-Kac Theorem (Aaronson 1997), for any $f \in L^1(m_y), f \geq 0$, one has strong convergence, denoted by

$$\frac{S_n^{\tilde{T}}(f)}{a_n} \xrightarrow{\mathfrak{L}} C(y) m_y(f) Y_\alpha, \tag{7}$$

which means

$$\int_B g\left(\frac{S_n^{\tilde{T}}(f)(x,z)}{a_n}\right) h_y(x,z) d\mu_y(x,z) \to E[g(C(y) m_y(f) Y_\alpha)], \tag{8}$$

for any bounded and continuous function g and for any probability density function h_y of $(B, \mathcal{B}, m_y)$. Here Y_α has the normalized Mittag-Leffler distribution of order $\alpha = 1 - \beta$. The Convergence Of Local Times Theorem follows canonically observing that $C(y) m_y(f)$ is independent of y.

Definition

An integer-valued measurable function f is said to satisfy the L^∞ conditional local limit theorem at 0 if there exists a sequence $g_n \in \mathbb{R}$ of real constants such that

$$\lim_{n\to\infty} g_n =: g(0) > 0$$

and

$$\|B_n P_{T^n}(\mathbb{I}_{S_n=x}) - g_n\|_\infty$$

decreases exponentially fast.

This condition is essentially stronger than condition (1) holding in $L^\infty(\mu)$ and the convergence is exponentially fast.

Theorem (Almost sure distributional limit theorem for the local times)

Let f be an integer-valued function satisfying the L^∞ conditional local limit theorem at 0 with $B_n = n^\beta L(n)$ and slowly varying function $L(n)$ converging to $c > 0$. Moreover, assume that the following two conditions are satisfied: for some constants $K > 0$ and $\delta > 0$ and for all bounded Lipschitz continuous functions $g, F \in C_b(\mathbb{R})$ and $x \in \mathbb{Z}$ it holds that

$$cov\left(g(\ell_k), F \circ T^{2k})\right) \leq (\log\log k)^{-1-\delta} \quad \& \tag{9}$$

$$\sum_{n=1}^{\infty} \left| E\left(\mathbb{I}_{\{S_n = x\}} - \mathbb{I}_{\{S_n = 0\}}\right) \right| \leq K(1 + |x|^{\frac{\alpha}{1-\alpha}}), \quad (\alpha = 1 - \beta) \tag{10}$$

Theorem (continued)

Then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \frac{1}{k} \mathbb{I}_{\{\frac{\ell_k}{a_k} \le x\}} = M(x) \quad a.s. \tag{11}$$

is equivalent to

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \frac{1}{k} \mu\{\frac{\ell_k}{a_k} \le x\} = M(x), \tag{12}$$

where $M(x)$ is a cumulative distribution function.

Corollary (Gibbs-Markov maps)

The almost sure distributional limit theorem for the local times holds under the same setting as in the previous Corollary for Gibbs-Markov transformation.

It is because the L^∞ conditional local limit theorem in the sense of the above definition holds and the assumptions on the transfer operator are satisfied.

Example (β transformation)

Almost sure distributional limit theorem can be obtained for the β-transformation.

THANK YOU

The following proposition will be used in the proof of ASDLT For Local Times Theorem:

Proposition

$$Var\left(\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}g(\frac{\ell_k}{a_k})\right) = O\left((\log\log N)^{(-1-\delta)}\right)$$

for some $\delta > 0$, *as* $N \to \infty$, *where* g *is any bounded Lipschitz function with Lipschitz constant* 1.

Define the characteristic function operator $P_t : L^1(P) \to L^1(P)$ by $P_t f := P_T(e^{it\phi} f)$, which is the perturbation of P_T. By induction, $P_t^n f = P_{T^n}(e^{itS_n} f)$. Let the space $\mathcal{L}$ be the subspace in $L^1(P)$ of all functions with norm: $\|f\| := \|f\|_\infty + D_f$ where D_f is the Lipschitz constant of f. We assume that P_t acts on $\mathcal{L}$ and has the following properties:

- There exists $\delta > 0$, such that when $t \in C_\delta := [-\delta, \delta]$, P_t has a representation: $P_t = \lambda_t \pi_t + N_t$, $\pi_t N_t = N_t \pi_t = 0$, and the π_t is a one-dimensional projection generated by an eigenfunction V_t of P_t, i.e. $P_t V_t = \lambda_t V_t$. It implies that $P_t^n = \lambda_t^n \pi_t + N_t^n$.
- There exists constants K, K_1 and $\theta_1 < 1$ such that on C_δ, $\|\pi_t\| \leq K_1$, $\|N_t\| \leq \theta_1 < 1$, $|\lambda_t| \leq 1 - K|t|^d$.
- There exists $\theta_2 < 1$ such that for $|t| > \delta$, $\|P_t\| \leq \theta_2 < 1$.
- ϕ is Lipschitz continuous.

Conditions 9 and 10 can be proved under these assumptions. This also will complete the proof of the corollaries since Gibbs-Markov maps satisfy all the assumptions above. An example not satisfying the above condition can be derived for functions of the fractional Brownian motion as in Denker and Zheng 2019.
The proof of (9) illustrates the type of argument:

Proof of (9).

Let $g(\frac{\ell_k}{a_k}) := \int_\Omega g(\frac{\ell_k}{a_k})dP + \hat{g} := C_k + \hat{g}$, then by $P_T^{2k} = P + N^{2k}$,

$$
\begin{aligned}
& cov\left(g(\frac{\ell_k}{a_k}), F \circ T^{2k})\right) \\
= & \int_\Omega g(\frac{\ell_k}{a_k})(F \circ T^{2k})dP - \int_\Omega g(\frac{\ell_k}{a_k})dP \int_\Omega F \circ T^{2k} dP \\
= & \int_\Omega P_T^{2k}\left(g(\frac{\ell_k}{a_k})\right) FdP - \int_\Omega g(\frac{\ell_k}{a_k})dP \int_\Omega FdP \\
= & \int_\Omega N^{2k}(\hat{g})FdP \\
\leq & \|N^{2k}(\hat{g})\| \|F\|_1 \\
\leq & C\theta_1^{2k}\|g\| \\
\leq & C(\log\log(2k))^{-1-\delta}.
\end{aligned}
$$