The essential coexistence phenomenon in Hamiltonian dynamics

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Setting

Let $\mathcal{M}$ be a smooth compact Riemannian manifold.
Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be a diffeomorphism, preserving the volume μ
The Lyapunov exponent of a vector $v \in T_x\mathcal{M} \setminus \{0\}$ at $x \in \mathcal{M}$ is

$$\chi(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|df_x^n v\|,$$

provided the limit exists.

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provided the limit exists.

By Birkhoff ergodic theorem, the limit exists at almost every point with respect the volume μ.

Regular vs. chaotic motions

In the early stages the theory of dynamical systems were dominated by the study of regular dynamics including presence and stability of periodic motions, translations on surfaces, etc. Here we call a system regular if all the Lyapunov exponents are zero.

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In the early stages the theory of dynamical systems were dominated by the study of regular dynamics including presence and stability of periodic motions, translations on surfaces, etc. Here we call a system regular if all the Lyapunov exponents are zero.

Typical chaotic motions are attributed to hyperbolicity. So we call a system chaotic if all the Lyapunov exponents are nonzero up to a set of measure zero.

Definition of coexistence

Definition

A dynamical system $(\mathcal{M}, f, \mu)$ exhibits coexistence of regular and chaotic behavior if $\mathcal{M} = \mathcal{R} \uplus \mathcal{C}$, where

- (i) $\chi(x, v) = 0 \quad \forall v \in T_x\mathcal{M}, x \in \mathcal{R}$;
- (ii) $\chi(x, v) \neq 0 \quad \forall v \in T_x\mathcal{M}, x \in \mathcal{C}$;
- (iii) $f|_{\mathcal{C}}$ is ergodic;
- (iv) $\mu(\mathcal{R}) > 0$ and $\mu(\mathcal{C}) > 0$.

Definition

A dynamical system $(\mathcal{M}, f, \mu)$ exhibits essential coexistence of regular and chaotic behavior if it exhibits coexistence of regular and chaotic behavior, and in addition

- (v) $\mathcal{C}$ is a dense subset.

Remarks

If restriced to $\mathcal{C}$, f is ergodic and does not have zero Lyapunov exponent, then f is Bernoulli by Pesin theory.
Hence, f is mixing with respect to the conditional measure on $\mathcal{C}$.
If (v) holds, then $\mathcal{C}$ is open and dense. So f topologially mixing.

Motivation 1 - Sinai's Conjecture

The Chirikov-Taylor standard map $f_\lambda : \mathbb{T}^2 \to \mathbb{T}^2$, for $\lambda \in \mathbb{R}$:

$$f_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \sin 2\pi(x+y) \end{pmatrix}.$$

- f_0 is a sheer map - completely integrable;
- $h_{top}(f_\lambda) > 0$ for $\lambda \neq 0$;
- each f_λ preserve the area μ.

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Conjecture (Sinai)

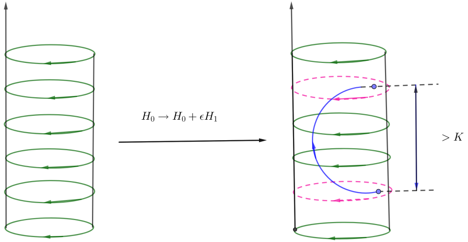
For any $\lambda \neq 0$, the metric entropy $h_\mu(f_\lambda) > 0$.

- By Pesin's entropy formula, Sinai's Conjecture $\Longleftrightarrow$ $\mu(\mathcal{C}) > 0$;
- If λ is small, Sinai Conjecture $\Longleftrightarrow$ the coexistence of KAM circles $\mathcal{R}$ and chaotic sea $\mathcal{C}$.

Motivation 2 - KAM and Arnold's diffusion

- KAM: Consider the complete integrable Hamiltonian system

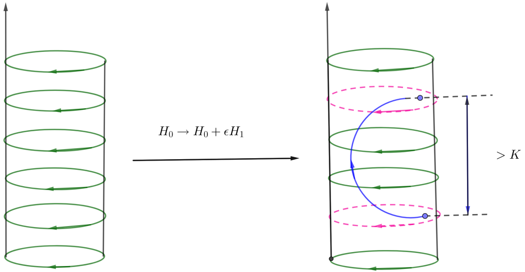
$$\begin{cases} \dot{\theta} = I, \\ \dot{I} = 0, \end{cases} \qquad \text{for } (\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n.$$



- Under small perturbation $H_0 = \frac{1}{2}I^2 \longrightarrow H_0 + \epsilon H_1$, a Cantor set $\mathcal{R}$ of non-resonant KAM invariant tori survives.

Motivation 2 - KAM and Arnold's Diffusion

- Arnold's diffusion: Assume $n \geq 3$. $\forall K > 0$, $\forall \epsilon \ll 1$, there exists an orbit γ such that
$$I(\gamma(t)) - I(\gamma(0)) > K \quad \text{for some } t > 0.$$
It suggest that $\mathcal{C} \supset \{x : \chi(x, \partial_I) \neq 0\}$ may have positive measure.



Volume-Preserving KAM

KAM also holds in volume-preserving category (Cheng-Sun, Herman, Xia, Yoccoz in 1990's). Consider

$$f_\epsilon \begin{pmatrix} \theta \\ I \end{pmatrix} = \begin{pmatrix} \theta + \omega(I) \\ I \end{pmatrix} + \epsilon F(\theta, I), \text{ for } (\theta, I) \in \mathbb{T}^n \times \mathbb{R},$$

where F is real-analytic, and is chosen such that f_ϵ preserves the volume.

Theorem

Under certain twist condition, for any small $\epsilon > 0$, a Cantor set $\mathcal{R}$ of invariant tori survives.

A question

The invaint tori are often called elliptic islands.

Question

What happens outside the elliptic islands? It the motion chaotic?

Or, equivalently,

Question

Are the elliptic islands surrounded by chaotic sea?

Arnold diffusion

There exist solutions to nearly integrable Hamiltonian systems that exhibit a significant change in the action variables.

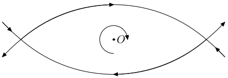
The purpose of the project is to understand the phenomena from smooth dynamical system point of view.

Coexistence in $\mathbb{T}^2$

Przytycki (1982) considered a family of area preserving diffeomorphisms $f_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$ given by

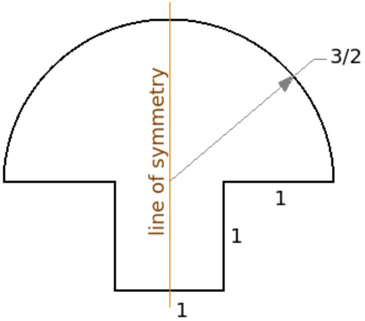
$$H_\varepsilon(x, y) = (x + y, y + h_\varepsilon(x + y))$$

for certain $h_\varepsilon : \mathbb{S}^1 \to \mathbb{S}^1$.
It is proved that one can choose a family h_ε to ensure that for every $\varepsilon < 0$ small the elliptic island O_ε is the domain between separatrices, and the map H_ε behaves stochastically on $\mathcal{C}_\varepsilon := \mathbb{T}^2 \backslash \overline{O_\varepsilon}$, more exactly, the Lyapunov exponents for $H_\varepsilon | \mathcal{C}_\varepsilon$ are nonzero almost everywhere and $H_\varepsilon | \mathcal{C}_\varepsilon$ is isomorphic to a Bernoulli automorphism.



Bunimovich mushrooms

Bunimovich (2008) constructed a biliard table consisting of a rectangle and a semicircle. In the system the orbits staying the semecircle form an invariant open set in the phase space.



Other results

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R. Devaney, A piecewise linear model for the zones of instability of an area preserving map, (1984)
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J.-M. Strelcyn, The "coexistence problem" for conservative dynamical systems: a review, (1991)

Essential coexistence in volume-preserving category

Theorem (H-Pesin-Talitskaya, 2013)

There exist diffeomorphisms $(\mathcal{M}^5, f, \mathrm{vol})$ displaying essential coexistence, such that $f|_{\mathcal{C}}$ is Bernoulli.

Theorem (Chen-H-Pesin, 2014)

There exist flows $(\mathcal{M}^5, f^t, \mathrm{vol})$ displaying essential coexistence, such that $f^t|_{\mathcal{C}}$ is Bernoulli.

Theorem (Jianyu Chen, 2013)

There exist diffeomorphisms $(\mathcal{M}^4, f, \mathrm{vol})$ displaying essential coexistence, such that $f|_{\mathcal{C}}$ has countably infinitely many ergodic components.

Main results

Theorem A (Chen-H-Pesin-Zhang, 2019)

There exist a four dimensional manifold $\mathcal{M}$ and a Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$ such that restricted to any energy surface $\mathcal{M}_e := \{H = e\}$, the Hamiltonian flow $f^t : \mathcal{M}_e \to \mathcal{M}_e$ exhibit essential coexistence. More precisely,

1. *There is an open and dense set $U = U_e$ such that $f^t(U) = U$ for any $t \in \mathbb{R}$, and $m(U^c) > 0$, where $U^c = \mathcal{M}_e \setminus U$ is the complement of U.*
2. *Restricted to U, $f^t|_U$ is hyperbolic and ergodic. In fact, $f^t|_U$ is Bernoulli.*
3. *Restricted to U^c, all orbits of $f^t|_{U^c}$ are periodic with zero Lyapunov exponents.*

Essential coexistence for a flow on $\mathbb{T}^3$

In order to construct the Hamiltonian flow, we first construct a 3-dimensional flow with essential coexistence.

Theorem B

There exists a volume preserving C^∞ flow f^t on $\mathcal{M} = \mathbb{T}^3$ that demonstrates the essential coexistence phenomenon, i.e., it has Property (1)-(3) in Theorem A, that is

(1) *there is an open and dense set $U \subset \mathbb{T}^3$ such that $f^t(U) = U$ for any $t \in \mathbb{R}$, and $m(U^c) > 0$, where $U^c = \mathcal{M} \setminus U$ is the complement of U;*

(2) *restricted to U, $f^t|_U$ is hyperbolic and ergodic, in fact, $f^t|_U$ is Bernoulli;*

(3) *restricted to U^c, all orbits of $f^t|_{U^c}$ are periodic with zero Lyapunov exponents.*

Essential coexistence for a diffeomorphism on $\mathbb{T}^2$

In order to construct the flow, we construct a diffeomorphism of $\mathbb{T}^2$ with essential coexistence.

Theorem C

There exists a C^∞ area preserving diffeomorphism f on $\mathbb{T}^2$ such that

(1) *there is an open and dense set $U \subset \mathbb{T}^2$ such that $f(U) = U$ and $m(U^c) > 0$, where $U^c = \mathbb{T}^2 \setminus U$ is the complement of U;*

(b) *restricted to U, $f|U$ is hyperbolic and ergodic; in fact, $f|U$ is Bernoulli;*

(c) *restricted to U^c, $f|U^c = \mathrm{id}$.*

Katok's map

Our construction is based on the Katok's map $g : \mathbb{D}^2 \to \mathbb{D}^2$.

Proposition 1

There is a C^∞ area-preserving diffeomorphism $g : \mathbb{D}^2 \to \mathbb{D}^2$ with the following properties:

1. *g is ergodic, and in fact, is isomorphic to a Bernoulli map;*
2. *g has non-zero Lyapunov exponents almost everywhere;*
3. *near $\partial\mathbb{D}^2$, g is the time-1 map of the flow generated by a vector field Z;*
4. *the map g can be constructed to be arbitrary flat near the boundary of the disk.*

Remark

The last part means that g is sufficiently close to the identity map. Hence, g can be extend to a C^∞ map $g : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ s.t. $g|_{\partial\mathbb{D}^2} = \mathrm{id}$.

Embedding of $\mathbb{D}^2$ into $\mathbb{T}^2$ with fractal boundary

Proposition 2

There exists a C^∞ diffeomorphism h from $\mathbb{D}^2$ into $\mathbb{T}^2$ with the following properties:

1. *the image $U = h(\mathbb{D}^2)$ is a simply connected, open and dense subset of $\mathbb{T}^2$. Moreover, $\partial U = \mathbb{T}^2 \setminus U = E \cup L$, where E is a Cantor set of positive Lebesgue measure and L is a union of countably many line segments;*
2. *h is area-preserving, i.e., $h^* m_U = m_{\mathbb{D}^2}$, where m_U is the normalized Lebesgue measure on U;*
3. *h can be continuously extended to $\partial\mathbb{D}^2$ such that $h(\partial\mathbb{D}^2) \subseteq \partial U$, and therefore for any $\varepsilon > 0$, $\mathcal{N}_\varepsilon = h^{-1}(V_\varepsilon)$ is a neighborhood of $\partial\mathbb{D}^2$, where $V_\varepsilon := \{x \in U : d(x, \partial U) < \varepsilon\}$.*

Proof of Proposition 2, Step 1: Construction of U

Let U_1 be a $(\alpha, 1)$ cross inscribed in $\mathbb{T}^2 = [0,1]^2$.
L_1 consists of the left/right edges and top/bottom edges U_1.
The four squares form E_1, denote by $E_{1,j}$, $j = 1, 2, 3, 4$.

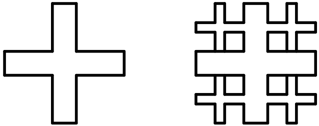

FIGURE 1. The construction of the sets U_1 and U_2

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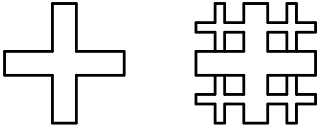


FIGURE 1. The construction of the sets U_1 and U_2

Put a cross inscribed in each $E_{1,j}$ and let U_2 be the union.
L_2 consists of the boundary of U_2.
The 4^4 squares form E_2.

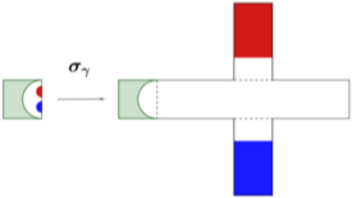
Fiinally, let $U = \bigcup_{n\geq 0} U_n$, $E = \bigcap_{n\geq 0} E_n$, $L = \bigcup_{n\geq 1} L_n$.

Step 2: Construction of a map $\varphi : \mathbb{D}^2 \to U$

Choose a C^ω diffeo. $\widehat{\varphi}_0 : \mathbb{D}^2 \to U_1$ by Riemann mapping theorem. For any $n \geq 1$ choose C^∞ map $\widehat{\varphi}_n : U_n \to U_{n+1}$ by

$$\widehat{\varphi}_n(x) = \begin{cases} \sigma_{\gamma_n}, & \text{if } x \in W_{n,k}, k = 1, \ldots, 4^n; \\ x, & \text{elsewhere,} \end{cases}$$

where



Then define

$$\varphi_n = \widehat{\varphi}_{n-1} \circ \cdots \circ \widehat{\varphi}_1 \circ \widehat{\varphi}_0, \quad \varphi = \lim_{n\to\infty} \varphi_n.$$

ϕ is a C^∞ map and can be continuously extended to $\partial\mathbb{D}^2$.

Step 3: Construction of a map $\psi : \mathbb{D}^2 \to \mathbb{D}^2$

We construct C^∞ diffeomorphism ψ on $\mathbb{D}^2$ such that $h := \varphi \circ \psi$ is area preserving and can be continuously extended to $\mathbb{D}^2$.
Let m_U be the normalized Leb. measure on U. Define $\mu = \varphi^* m_U$.
We construct a map ψ such that $\psi^* \mu = m_{\mathbb{D}^2}$.
This can be obtained by using Moser's theorem.

Theorem (Moser)

Given two smooth volume forms μ and ν with the same total volume on a compact connected manifold M, there exists a diffeomorphism ϕ s.t. $\mu = \phi^* \nu$.

Step 3: Construction of a map $\psi : \mathbb{D}^2 \to \mathbb{D}^2$

Since $\mathbb{D}^2$ is not compact, and the density of μ is unbounded, we use the theorem to construct a sequence diffeomorphisms $\{\widehat{\psi}_n\}$ such that $\widehat{\psi}_n|\varphi^{-1}(U_{n-1}) = \mathrm{id}$ and $d(x, \widehat{\psi}_n(x)) \to 0$ uniformly as $n \to \infty$.

Then define

$$\phi_n = \widehat{\psi}_{n-1} \circ \cdots \circ \widehat{\psi}_1, \quad \psi = \lim_{n\to\infty} \psi_n.$$

ϕ is a C^∞ map and can be continuously extended to $\partial\mathbb{D}^2$.
Finally let $h = \varphi \circ \psi$, which is the required map.

Proof of Theorem C

Theorem C

There exists a C^∞ area preserving diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ that display essential coexistence.

Proof.

Take $g : \mathbb{D}^2 \to \mathbb{D}^2$ as in Proposition 1 (Katok's map).
Take $h : \mathbb{D}^2 \to U \subsetneqq \mathbb{T}^2$ as in Proposition 2.

Define $f(x) = \begin{cases} (h \circ g \circ h^{-1})(x), & x \in U; \\ \mathrm{id}, & \text{elsewhere.} \end{cases}$

Choose g "flat" enough near $\partial\mathbb{D}^2$ such that f is C^∞ on ∂U. □

Isotopy from identity to Katok's map

Using Smale's isotopy theorem on $\mathbb{D}^2$ we can get,

Proposition 3

Then there is a C^∞ map $G : \overline{\mathbb{D}^2} \times [0,1] \to \overline{\mathbb{D}^2}$ such that

1. *for any $t \in [0,1]$ the map $g_t = G(\cdot, t) : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ is an area-preserving diffeomorphism;*
2. *$g_0 = id$ and $g_1 = g$;*
3. *$d^n G(x,1) = d^n G(g(x),0)$ for any $n \geq 0$;*
4. *in a neighborhood $\mathcal{N}$ of $\partial\mathbb{D}^2$, $g_t|\mathcal{N}$ is the flow generated by Z;*
5. *for each $t \in [0,1]$ the map g_t can be constructed to be arbitrary flat near the boundary of the disk.*

Proof of Theorem B

Theorem B

There exists a C^∞ volume preserving flow $f^t : \mathbb{T}^3 \to \mathbb{T}^3$ that display essential coexistence.

Proof.

- Define $F : \mathbb{T}^3 \to \mathbb{T}^2$ by $F(x,t) = \begin{cases} h \circ G(h^{-1}(x), t), & x \in U, \\ id, & \text{elsewhere.} \end{cases}$
  and denote $\widetilde{f}_t = F(\cdot, t) = h \circ g_t \circ h^{-1}$ on the suspension manifold $\mathcal{K}$.
- Note $\hat{F} : \mathcal{K} \to \mathbb{T}^3$ given by $\hat{F}(x,\theta) = (F(x,\theta), \theta)$ is well-defined, then we get a flow $\hat{f}_t = \hat{F} \circ \widetilde{f}_t \circ \hat{F}^{-1}$, whose vector field is $\hat{X}(x,\theta) = (X_1(x_1,x_2,\theta), X_2(x_1,x_2,\theta), 1)$.
- Make a time change to get an ergodic flow with vector field $X(x,\theta) = (X_1(x_1,x_2,\theta), X_2(x_1,x_2,\theta), \tau(x_1,x_2))$.

Proof of Theorem A

Theorem A

There exists a 4-dim manifold $\mathcal{M}$ and a Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$ s. t. the Hamiltonian flow $f^t : \mathcal{M}_e \to \mathcal{M}_e$ exhibit essential coexistence. on the energy surface $\mathcal{M}_e = \{H = e\}$.

Idea of Proof.

- Define $\mathcal{M} = \mathbb{T}^3 \times \mathbb{R} =: \{(x_1, x_2, \theta, I)\}$.

  There is a standard symplectic form $\omega = dx_1 \wedge dx_2 + d\theta \wedge dI$.

- Let $X = (X_1, X_2, \tau)$ be the vector field given in Theorem B that generates the flow f^t in $\mathbb{T}^3$.

  Let $\Theta = \Theta(x_1, x_2, \theta)$ be the backward hitting time to the zero level for f^t initiated at $(x_1, x_2, \theta) \in \mathbb{T}^3$, that is, $\exists!(\widehat{x}_1, \widehat{x}_2)$ and a unique value $\Theta \in [0, 1) \cong \mathbb{T}$ s. t. $f^\Theta(\widehat{x}_1, \widehat{x}_2, 0) = (x_1, x_2, \theta)$.

  Define $\Phi : \mathcal{M} \to \mathcal{M}$ by $\Phi(x_1, x_2, \theta, I) = (x_1, x_2, \Theta, I)$. Then define $\hat{\omega} = \Phi^*\omega = dx_1 \wedge dx_2 + d\Theta \wedge dI$.

Proof of Theorem A

Idea of Proof (Construction of the Hamiltonian function).

- Let $\widetilde{H} = \widetilde{H}(x_1, x_2, \theta) : \mathbb{T}^3 \to \mathbb{R}$ be the function given by the equation

$$\begin{cases} \dfrac{\partial \widetilde{H}}{\partial x_2} = X_1, \ -\dfrac{\partial \widetilde{H}}{\partial x_1} = X_2, & \text{on } U, \\ \widetilde{H} = 0 & \text{on } \partial U, \end{cases}$$

  Let $\widehat{H}(x_1, x_2, \theta, I) = \widetilde{H}(x_1, x_2, \Theta(x_1, x_2, \theta)) + I.$

  Finally, set $H = \Phi_* \widehat{H} = \widehat{H} \circ \Phi^{-1}$ and $X_H = \Phi_* X_{\widehat{H}}$.

- Clearly, $\mathcal{M}_e = \{H = e\} = \Phi\{\widehat{H} = e\} = \Phi\widehat{\mathcal{M}}_e$.

  The vector field is given by $X_H = \Phi_* X_{\widehat{H}}$, where

  $X_{\widehat{H}} = \left(X_1, X_2, \tau, \frac{\partial \widetilde{H}}{\partial \theta}\right)$.

□

Thank you!