Rigidity and Classification of Cantor Actions

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In this talk, describe program of research that combines ideas from the programs at the M.S.R.I. during the *excellent* years 1983-85.

Problem: Classify weak solenoids, up to homeomorphism.

Solution: Classify arboreal actions of finitely-generated groups, up to return equivalence.

- If G is Noetherian group $\Rightarrow$ *Rigidity Property*.
- If G admits uncountably many subgroups $\Rightarrow$ *Wild Actions*.
- Related to properties of *Invariant Random Subgroups*.
- Applications to *Arithmetic Number Theory*.
- Variety of *Open Problems*.

M_0 a compact manifold without boundary.

Given a sequence of proper non-trivial coverings, set

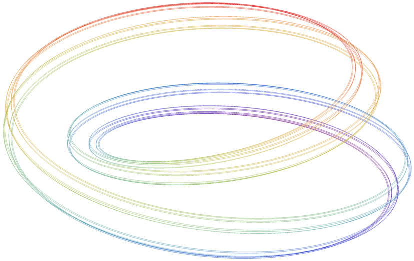
$$\begin{aligned} \mathfrak{M} = \varprojlim \; & \{p^{\ell}_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \\ = \;& \{(y_0, y_1, y_2, \ldots) \mid p^{\ell}_{\ell+1}(y_{\ell+1}) = y_\ell \mid \ell \geq 0\} \\ \subset \;& \prod_{\ell \geq 0} M_\ell \end{aligned}$$

- $\mathfrak{M}$ is a *(weak) solenoid*, with foliation $\mathcal{F}_{\mathfrak{M}}$.
- Leaves are the path connected components of $\mathfrak{M}$, which are non-compact covering spaces of M_0
- $(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$ is a *generalized lamination* $\equiv$ *foliated space* with Cantor transversals.

Everybody's favorite example: Vietoris solenoids

$$\mathbb{S}^1 \xleftarrow{m_1} \mathbb{S}^1 \xleftarrow{m_2} \mathbb{S}^1 \xleftarrow{m_3} \mathbb{S}^1 \xleftarrow{m_4} \mathbb{S}^1 \cdots$$

Each $m_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1$ is an m_ℓ-fold covering map, $m_\ell > 1$.



- $G = \pi_1(M, x_0)$ is finitely generated group.

$$M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} M_3 \cdots$$

Choose $x_\ell \in M_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(M_\ell, x_\ell)$

Inclusion maps $q_{\ell+1} \colon G_{\ell+1} \subset G_\ell$, descending chain of groups

$$G = G_0 \xleftarrow{q_1} G_1 \xleftarrow{q_2} G_2 \xleftarrow{q_3} G_3 \cdots$$

$$\mathfrak{X} = \varprojlim \, \{G_0/G_{\ell+1} \longrightarrow G_0/G_\ell\}$$

The left G action Φ on Cantor set $\mathfrak{X}$ is conjugate to monodromy action on transversal in $\mathfrak{M}$. Action is minimal and equicontinuous.

- $(\mathfrak{X}, G, \Phi)$ is a Cantor action if $\mathfrak{X}$ is Cantor set, and action is minimal and equicontinuous. This is equivalent to an action of G on a pointed tree, or arboreal action.
- Clopen set $U \subset \mathfrak{X}$ is adapted if the stabilizer is a subgroup

$$G_U = \{g \in G \mid \varphi(g)(U) = U\}$$

- G_U has finite index, and acts transitively on the finite set of translates $\{g \cdot U \mid g \in G\}$ (the vertices in a tree model)
- $\mathcal{H}_U^{\Phi} \equiv \{\Phi(g)|U \mid g \in G_U\} \subset \mathbf{Homeo}(U)$.

Definition: Equicontinuous Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$ are *return equivalent* if there exists adapted sets $U \subset \mathfrak{X}$ and $V \subset \mathfrak{Y}$ and a homeomorphism $h \colon U \to V$ which conjugates the groups $\mathcal{H}_U^{\Phi} \subset \mathbf{Homeo}(U)$ and $\mathcal{H}_V^{\Psi} \subset \mathbf{Homeo}(V)$.

Theorem: Given weak solenoids $\mathfrak{M}$ and $\mathfrak{M}'$ with monodromy actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$, if $\mathfrak{M}$ and $\mathfrak{M}'$ are homeomorphic, then their monodromy actions are return equivalent.

There is a not a converse to this, in general, except in special cases:

Definition: $\mathfrak{M}$ is a nil-solenoid if the base manifold M_0 is a compact nil-manifold, and $\mathcal{F}_{\mathfrak{M}}$ has a simply connected leaf.

We then have a generalization of the classification result for 1-dimensional solenoids by Aarts and Fokkink.

Theorem: Let $\mathfrak{M}$ and $\mathfrak{M}'$ be nil-solenoids. If their monodromy actions are return equivalent, then the spaces are homeomorphic.

- Classify Cantor actions up to return equivalence.
- Give invariants of return equivalence.

Definition: Φ is *locally quasi-analytic* (LQA) if there exists $\epsilon > 0$ so that if U adapted and $\mathrm{diam}_{\mathfrak{X}}(U) < \epsilon$, for all clopen $V \subset U$,

$$\Phi(g)|V = Id \implies \Phi(g)|U = Id \quad , \quad \text{for all } g \in G_U$$

That is, the action of $\mathcal{H}_U^{\Phi}$ on U is *topologically free*.

Definition: Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$ are *continuously orbit equivalent* (COE) if there exists a homeomorphism $h \colon \mathfrak{X} \to \mathfrak{Y}$ and continuous functions

$$\alpha \colon G \times \mathfrak{X} \to H, \; h(\Phi(g)(x)) = \Psi(\alpha(g, x), h(x)), \; g \in G, x \in \mathfrak{X}$$

$$\beta \colon H \times \mathfrak{Y} \to G, \; h^{-1}(\Psi(g, y)) = \Phi(\beta(g, y), h^{-1}(y)), \; g \in H, y \in \mathfrak{Y}$$

Actions are *locally continuously orbit equivalent* (LCOE) if there exists adapted subsets $U \subset \mathfrak{X}$ and $V \subset \mathfrak{Y}$ such that the restricted actions are continuously orbit equivalent.

- Renault showed that LCOE is basic notion for isomorphism of cross-product C^* -algebras with Cartan subalgebra.

- Extend Cortez & Medynets *rigidity theorem* for free actions:

Theorem: Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$ be LQA Cantor actions.

Locally Continuously Orbit Equivalent $\Leftrightarrow$ Return Equivalent

Definition: G is *Noetherian* if every subgroup $G' \subset H$ is finitely generated.

Theorem: G Noetherian $\Rightarrow$ every Cantor action of G is LQA.

Corollary: Cantor actions by Noetherian groups satisfy orbit equivalence rigidity.

Definition: $\mathfrak{G}(\Phi) = \overline{H_\Phi}$ is closure of $H_\Phi = \Phi(G) \subset \mathbf{Homeo}(\mathfrak{X})$ in the *uniform topology on maps*.

$\mathfrak{G}(\Phi)$ is a profinite group acting transitively on $\mathfrak{X}$.

$\mathcal{D}_x = \mathfrak{G}(\Phi)_x = \{\widehat{h} \in \overline{H_\Phi} \mid \widehat{h} \cdot x = x\}$ (isotropy group of $x \in \mathfrak{X}$)

$\mathcal{D}_x \subset \mathfrak{G}(\Phi)$ is independent of the choice of basepoint x, up to topological isomorphism.

$\mathcal{D}_x$ acts effectively on $\mathfrak{X}$.

The action $(\mathfrak{X}, \mathfrak{G}(\Phi), \widehat{\Phi})$ is called the profinite model for $(\mathfrak{X}, G, \Phi)$.

Proposition: $\mathcal{D}_x$ is *totally not normal*: for any $\widehat{h} \in \mathcal{D}_x$ there exists $\widehat{g} \in \mathfrak{G}(\Phi)$ such that $\widehat{g}^{-1}\ \widehat{h}\ \widehat{g} \notin \mathcal{D}_x$.

For group chain $\mathcal{G} = \{G_\ell \mid \ell \geq 0\}$ the normal core of G_ℓ in G

$$C_\ell = \bigcap_{g \in G} g G_\ell g^{-1} \subset G_\ell$$

Theorem [Dyer-Hurder-Lukina, 2016].

$$\mathcal{D}_x \cong \varprojlim\ \{\pi_{\ell+1} \colon G_{\ell+1}/C_{\ell+1} \to G_\ell/C_\ell \mid \ell \geq 0\}\ .$$

- This result is abstract, but in practice is often effective for calculating the discriminant group $\mathcal{D}_x$ for a given group chain.

Recipe for Cantor actions:

- ⋆ Take one finitely-generated group G.
- ⋆ Choose a profinite completion $\mathfrak{G}(\Phi)$ of G.
- ⋆ Choose totally not normal closed subgroup $\mathcal{D} \subset \mathfrak{G}(\Phi)$.

Then action of G on $\mathfrak{X} \equiv \mathfrak{G}(\Phi)/\mathcal{D}$ is minimal and equicontinuous.

1. $\mathcal{D}_x$ is trivial for Cantor action $(\mathfrak{X}, G, \Phi)$ with G abelian.
2. $\mathcal{D}_x$ can be a Cantor group for a Cantor action $(\mathfrak{X}, G, \Phi)$ when G is 3-dimensional Heisenberg group.
3. Every finite group and every separable profinite group can be realized as $\mathcal{D}_x$ for a Cantor action by a torsion-free, finite index subgroup of $\mathbf{SL}(n, \mathbb{Z})$.
4. $\mathcal{D}_x$ can be wide-ranging for arboreal representations of absolute Galois groups of number fields and function fields.
5. Every Cantor action by a finitely generated group G can be realized as the monodromy of a weak solenoid.

The proof of 3 uses ideas of **Lubotzky** on torsion elements in the profinite completion of torsion free subgroups of $\mathbf{SL}(n, \mathbb{Z})$, and a construction due to **Lenstra**.

Problem: If G is not Noetherian, then how to classify its actions?

Let $(\mathfrak{X}, G, \Phi)$ be a Cantor action which is not LQA.

Choose an *adapted neighborhood basis* $\{U_\ell \mid \ell \geq 1\}$ of $x \in \mathfrak{X}$.

Set $G_0 = G$, $G_\ell = G_{U_\ell}$ for $\ell \geq 1$.

Then $\mathcal{G} = \{G_\ell \mid \ell \geq 0\}$ defines $\mathfrak{X}$ as inverse limit space.

Set $K_\ell = \{g \in G \mid \Phi(g)|U_\ell = Id\}$.

$K_\ell \subset K_{\ell+1}$ so $\mathfrak{K}^x = \{K_\ell \mid \ell \geq 1\}$ is increasing chain.

Theorem: $(\mathfrak{X}, G, \Phi)$ is LQA $\Leftrightarrow \{K_\ell \mid \ell \geq 1\}$ is bounded.

Theorem: If Cantor action $(\mathfrak{X}, G, \Phi)$ is not LQA, then G admits uncountably many subgroups.

Sketch of proof:

Let $\mathfrak{G} = \{U_\ell \mid \ell \geq 1\}$ be the adapted neighborhood basis of $x \in \mathfrak{X}$.

For any $y \in \mathfrak{X}$, there is $\widehat{g} \in \mathfrak{G}(\Phi)$ with $\widehat{g} \cdot x = y$.

$\widehat{g} = (g_0, g_1, g_2, \ldots)$ with $g_\ell \in G$ and $g_\ell C_\ell = g_{\ell+1} C_\ell$

$C_\ell \subset G_\ell$ is the normal core subgroup.

Then $\mathfrak{G}^y = \{g_\ell \cdot U_\ell \mid \ell \geq 1\}$ is adapted neighborhood basis of y.

$\mathfrak{K}^y = \{K_\ell^y = g_\ell \; K_\ell \; g_\ell^{-1} \mid \ell \geq 1\}$ is increasing subgroup chain.

$K_\infty^y = \bigcup_{\ell \geq 1} \; K_\ell^y$ is infinitely generated if $\mathfrak{K}^x$ is not bounded, and $K_\infty^y \neq K_\infty^z$ if $y \neq z$.

Bartholdi, Grigorchuk, Nekrashevych, et al: Examples of weakly branch group actions on trees induce non-LQA actions on the Cantor boundary of a d-ary tree, $d \geq 2$.

Lukina: Let $f(x)$ be a quadratic polynomial with critical point c. If the post-critical set P_C contains at least 3 points, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ and $\mathrm{Gal}_{\mathrm{arith}}(f)$ on the boundary of the tree formed by iterated solutions is non-LQA.

Groeger & Lukina: If Cantor action $(\mathfrak{X}, G, \Phi)$ is not LQA, then the push-forward of an ergodic measure on $\mathfrak{X}$ via the mapping $x \mapsto G_x$ is a continuous (non-atomic) I.R.S. This is a broader class of examples than just weakly branch actions.

- Thus, can use *Ergodic Theory/Descriptive Set Theory* to classify non-LQA Cantor actions.

There is another approach to the study of non-LQA Cantor actions.

Recall: $\mathfrak{G}(\Phi) = \overline{H_\Phi} \subset \mathbf{Homeo}(\mathfrak{X})$

$\mathcal{D}_x = \mathfrak{G}(\Phi)_x$ (isotropy group of $x \in \mathfrak{X}$)

$\mathcal{D}_x \subset U$ for all adapted $x \in U \subset \mathfrak{X}$

$\mathfrak{G} = \{U_\ell \mid \ell \geq 1\}$ adapted neighborhood basis of $x \in \mathfrak{X}$.

$\Longrightarrow \quad \mathcal{D}_x \subset U_\ell$ for all $\ell \geq 1$.

$\widehat{\Phi}_\ell \colon G_\ell \times U_\ell \to U_\ell$ induces a local action map

$$\rho_\ell \colon \mathcal{D}_x \to \mathbf{Homeo}(U_\ell)$$

Set $\widehat{K}_\ell \equiv \ker\{\rho_\ell\} \subset \mathfrak{G}(\Phi)$ for $\ell \geq 1$. Then $\widehat{K}_1 \subset \widehat{K}_2 \subset \cdots$

Theorem: The isomorphism class of the direct limit group

$$\Upsilon(\Phi) = \varinjlim \{\widehat{K}_\ell \subset \widehat{K}_{\ell+1} \mid \ell \geq 1\}$$

is a conjugacy invariant of a Cantor action $(\mathfrak{X}, G, \Phi)$.

A Cantor action $(\mathfrak{X}, G, \Phi)$ is:

- stable if the chain $\{\widehat{K}_\ell \mid \ell \geq 1\}$ is bounded.

That is, if there exists ℓ_0 so that $\widehat{K}_\ell = \widehat{K}_{\ell+1}$ for $\ell \geq \ell_0$.

- wild if the chain $\{\widehat{K}_\ell \mid \ell \geq 1\}$ is unbounded.

Theorem: The property that a Cantor action is wild, is a locally continuous orbit equivalence invariant.

The class of wild actions can be divided into two subclasses.

Let $\widehat{U}_\ell = \{\widehat{g} \in \mathfrak{G}(\Phi) \mid \widehat{\Phi}(\widehat{g}) \cdot U_\ell = U_\ell\}$
The collection $\{\widehat{U}_\ell \mid \ell \geq 1\}$ is a neighborhood basis of clopen sets about the identity $\widehat{e} \in \mathfrak{G}(\Phi)$. Consider the restricted Adjoint action

$$\widehat{\rho}_\ell \colon \mathcal{D}_x \to \mathbf{Homeo}(\widehat{U}_\ell)$$

Set $\widehat{K}^a_\ell \equiv \ker\{\widehat{\rho}_\ell\} \subset \mathfrak{G}(\Phi)$ for $\ell \geq 1$. Then $\widehat{K}^a_1 \subset \widehat{K}^a_2 \subset \cdots$
Observe that $\widehat{K}^a_\ell \subset \widehat{K}_\ell$ for all $\ell \geq 1$.

Analogous to homogeneous space $M = G/K$ where geometry of model M is studied via adjoint representation of K on the Lie algebra $\mathfrak{g}$ of G. Local model for Cantor action may be unstable.

Definition: A wild Cantor action $(\mathfrak{X}, G, \Phi)$ is:

- flat wild if $\widehat{K}^a_\ell = \widehat{K}_\ell$ for ℓ sufficiently large.
- dynamically wild if $\widehat{K}^a_\ell \neq \widehat{K}_\ell$ for ℓ sufficiently large.

- The Cantor actions associated to weakly branched groups are *dynamically wild*.

- There exists lattices $G \subset \mathbf{SL}_N(\mathbb{Z})$ with actions on a Cantor space $\mathfrak{X}$ that are *flat wild*.

- In the dynamically wild case, it is not known if the chain $\{\widehat{K}^a_\ell \mid \ell \geq 1\}$ must be unbounded as well.

Question 1: How to classify Cantor actions of a finitely-generated nilpotent group G?

Use invariants of the associated cross-product C^*-algebra?
In terms of the representations of G?

Question 2: If an action is wild, when is the action non-LQA?

Question 3: For which numbers fields and polynomials f is the action of the absolute Galois group $\mathrm{Gal}_{\mathrm{arith}}(f)$, on the boundary of the tree of iterated solutions, non-LQA?

Question 4: If G is a higher rank lattice and the action is effective, must it be stable?

Question 5: If G is a higher rank lattice and the action is wild, must it be flat wild?

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