Can you hear the shape of a drum ? and Deformational Spectral Rigidity

V. Kaloshin

August 15, 2019

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac ‘Can you hear the shape of a drum?’
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac 'Can you hear the shape of a drum?'
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac ‘Can you hear the shape of a drum?’
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac 'Can you hear the shape of a drum?'
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac ‘Can you hear the shape of a drum?’
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

- M. Kac 'Can you hear the shape of a drum?'
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac ‘Can you hear the shape of a drum?’
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac 'Can you hear the shape of a drum?'
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac ‘Can you hear the shape of a drum?’
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum? and Deformational Spectral Rigidity

- M. Kac ‘Can you hear the shape of a drum?’
- Laplace spectrum, Inverse problems
- Length spectrum and Laplace spectrum
- Spectral Rigidity for domains
- Spectral Rigidity for Anosov geodesic flow, Burns-Katok Conjecture
- Linearized Isospectral Operators and X-ray transforms

Can you hear the shape of a drum?

M. Kac'66: Can you hear the shape of a drum?

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0, \pi] \times [0, \pi] \ni (x, y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x, y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0, \pi] \times [0, \pi] \ni (x, y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x, y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2+m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area}(\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area}\,(\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0, \pi] \times [0, \pi] \ni (x, y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x, y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m\in\mathbb{Z}_+\setminus 0} \sqrt{k^2+m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda\to\infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area}\,(\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m\in\mathbb{Z}_+\setminus 0} \sqrt{k^2+m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda\to\infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area}\,(\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m\in\mathbb{Z}_+\setminus 0} \sqrt{k^2+m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda\to\infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0, \pi] \times [0, \pi] \ni (x, y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x, y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

Can you hear the shape of a drum?

Consider the Dirichlet problem in a domain $\Omega \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u + \lambda^2 u = 0 \\ u|_{\partial\Omega} = 0. \end{cases}$$

$\Delta(\Omega) := \{0 < \lambda_1 \le \lambda_2 \le \cdots\}$ — Laplace spectrum.

Example 1 Let $\Omega_C = [0,\pi] \times [0,\pi] \ni (x,y)$. For any pair $k, m \in \mathbb{Z}_+ \setminus 0$ let

$$u(x,y) = \sin kx \cdot \sin my \quad \text{and} \quad \lambda = \sqrt{k^2 + m^2}.$$

The Laplace spectrum $\Delta(\Omega_C) = \cup_{k,m \in \mathbb{Z}_+ \setminus 0} \sqrt{k^2 + m^2}$.

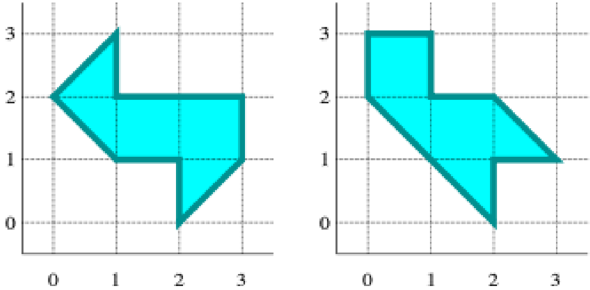
Question (M. Kac'66) Does $\Delta(\Omega)$ determine Ω up to isometry?

Weyl law (H. Weyl'11) $N(\lambda) := \#$ eigenvalues (w multiplicity) in $(0, \lambda^2]$, then

$$\lim_{\lambda \to \infty} \lambda^{-1} N(\lambda) = (4\pi)^{-1} \text{Area } (\Omega).$$

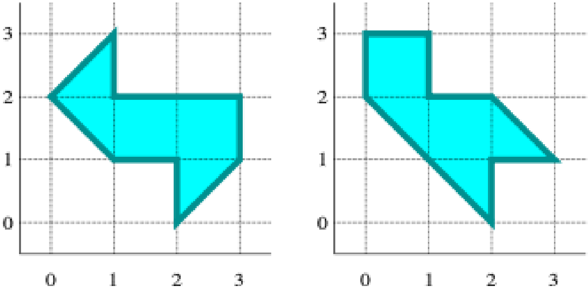
Can't hear the shape of a drum!

Gordon–Webb-Wolpert'92



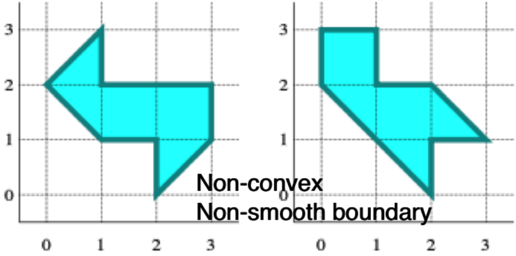
Can’t hear the shape of a drum!

Gordon–Webb-Wolpert’92



Can’t hear the shape of a drum!

Gordon–Webb-Wolpert’92



Consider domains with a smooth or an analytic boundary!

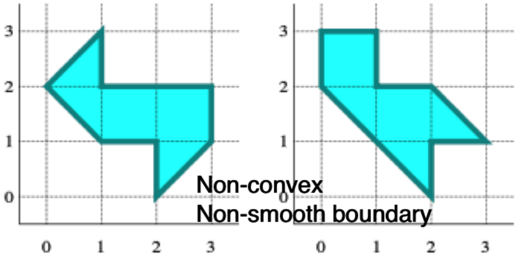
Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak’90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can’t hear the shape of a drum!

Gordon–Webb-Wolpert’92



Consider domains with a smooth or an analytic boundary!

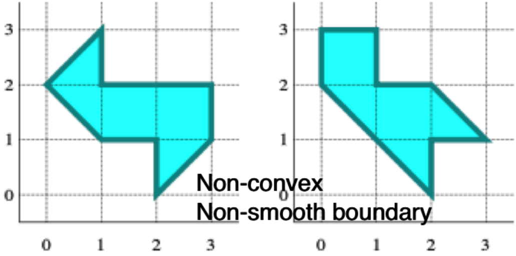
Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak’90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can’t hear the shape of a drum!

Gordon–Webb-Wolpert’92



Non-convex
Non-smooth boundary

Consider domains with a smooth or an analytic boundary!

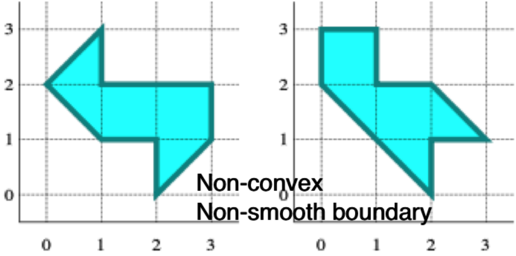
Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak’90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can't hear the shape of a drum!

Gordon–Webb-Wolpert'92



Consider domains with a smooth or an analytic boundary!

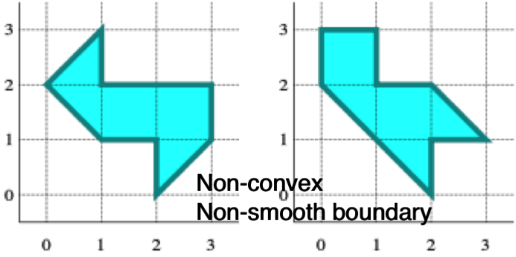
Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can’t hear the shape of a drum!

Gordon–Webb-Wolpert’92



Consider domains with a smooth or an analytic boundary!

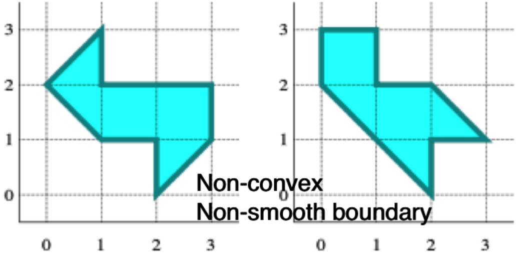
Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak’90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can't hear the shape of a drum!

Gordon–Webb-Wolpert'92



Consider domains with a smooth or an analytic boundary!

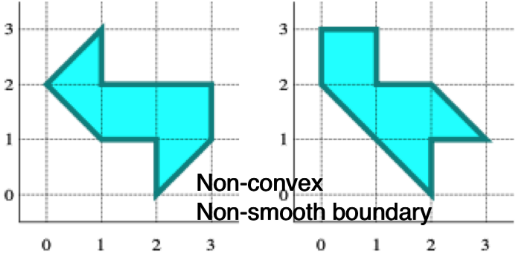
Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can't hear the shape of a drum!

Gordon–Webb-Wolpert'92



Consider domains with a smooth or an analytic boundary!

Osgood-Phillips-Sarnak A C^∞ isospectral set is compact.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Hezari-Zeldich, Popov-Topalov Analytic deformations of ellipses.
Colin de Verdière, Zeldich Generic analytic axis-symmetric domains.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Can you hear the shape of a Riemannian manifold?

Let (M, g) be a Riemannian compact manifold. Consider the spectrum of the Laplace-Beltrami operator $\Delta(M, g)$.

Question Does $\Delta(M, g)$ determine (M, g) up to an isometry?

Sunada, Vingeras* $\exists$ isospectral sets of arbitrary finite cardinality.

Conjecture (Sarnak'90) A C^∞ isospectr. set consists of isolated points.

Call Ω *spectrally rigid* (SR) if any smooth isopectral deformation $\{\Omega_t\}_t$ is an isometry, i.e. $\Delta(\Omega_t) \equiv \Delta(\Omega_0)$.

Conjecture (Sarnak'90) Any planar domain is SR.

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \quad \mathcal{L}(\Omega) := \cup_P L(P) \ \cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

$$\text{sing. supp. of } w(t) = \pm\mathcal{L}(\Omega) \cup 0.$$

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \quad \mathcal{L}(\Omega) := \cup_P L(P) \ \cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

sing. supp. of $w(t) = \pm\mathcal{L}(\Omega) \cup 0$.

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \quad \mathcal{L}(\Omega) := \cup_P L(P) \ \cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

$$\text{sing. supp. of } w(t) = \pm\mathcal{L}(\Omega) \cup 0.$$

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \;\; \mathcal{L}(\Omega) := \cup_P L(P) \;\cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

$$\text{sing. supp. of } w(t) = \pm\mathcal{L}(\Omega) \cup 0.$$

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \quad \mathcal{L}(\Omega) := \cup_P L(P) \ \cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

$$\text{sing. supp. of } w(t) = \pm\mathcal{L}(\Omega) \cup 0.$$

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \quad \mathcal{L}(\Omega) := \cup_P L(P) \ \cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

$$\text{sing. supp. of } w(t) = \pm\mathcal{L}(\Omega) \cup 0.$$

Length spectrum and Laplace spectrum

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. Define

$$\textit{the length spectrum} \quad \mathcal{L}(\Omega) := \cup_P L(P) \ \cup \mathbb{N}\, L(\partial\Omega),$$

$L(P)$ – perimeter of a periodic orbit, $\cup$ – over all per orbits.

Theorem (Chazarian, Anderson-Melrose, Guillemin, Duistermaat, ...) The Laplace $\Delta(\Omega)$ determines the length $\mathcal{L}(\Omega)$, generically. More exactly, the wave trace

$$w(t) = \mathrm{Re} \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t)$$

is C^∞ outside of $\pm\mathcal{L}(\Omega) \cup 0$. Generically,

$$\text{sing. supp. of } w(t) = \pm\mathcal{L}(\Omega) \cup 0.$$

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14, Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

A long standing conjecture of Burns-Katok

Let (M, g) be a smooth closed Riemannian compact manifold. Let **the length spectrum** of g be the set of lengths of closed geodesics. Let the metric g admit an Anosov geodesic flow. The closed geodesics are parametrized by the set $\mathcal{C}$ of free homotopy classes. Define **the marked length spectrum** by

$$\mathcal{L}_g : \mathcal{C} \to \mathbf{R}_+, \qquad L_g(c) := \ell_g(\gamma_c).$$

Conjecture [Burns-Katok'85] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the **same marked length spectrum**, i.e $L_g = L_{g_0}$, then they are **isometric**, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $g = g_0$.

Guillemin-Kazhdan'80, Katok'88 metrics in a fixed conformal class, Croke-Sharafutdinov'98, Dairbekov-Sharafutdinov'03, Paternain-Salo-Uhlmann'14,Guillarmou'17.

Guillarmou–Lefeuvre breakthrough

Theorem [GL'19] Let g be a negatively curved metric. Then for $N > \frac{3}{2}\dim M + 8$ there exists $\varepsilon > 0$ s. t. for any smooth metric g with same marked length spectrum as g_0 and s. t. $\|g - g_0\|_{C^N(M)} < \varepsilon$, then g and g_0 are isometric, i.e. there exists a diffeomorphism $\phi : M \to M$ s. t. $\phi^* g = g_0$.

Question (Sarnak) Is the same true for the length spectrum?

Guillarmou–Lefeuvre breakthrough

Theorem [GL'19] Let g be a negatively curved metric. Then for $N > \frac{3}{2} \dim M + 8$ there exists $\varepsilon > 0$ s. t. for any smooth metric g with same marked length spectrum as g_0 and s. t. $\|g - g_0\|_{C^N(M)} < \varepsilon$, then g and g_0 are isometric, i.e. there exists a diffeomorphism $\phi : M \to M$ s. t. $\phi^* g = g_0$.

Question (Sarnak) Is the same true for the length spectrum?

Guillarmou–Lefeuvre breakthrough

Theorem [GL'19] Let g be a negatively curved metric. Then for $N > \frac{3}{2}\dim M + 8$ there exists $\varepsilon > 0$ s. t. for any smooth metric g with same marked length spectrum as g_0 and s. t. $\|g - g_0\|_{C^N(M)} < \varepsilon$, then g and g_0 are isometric, i.e. there exists a diffeomorphism $\phi : M \to M$ s. t. $\phi^* g = g_0$.

Question (Sarnak) Is the same true for the length spectrum?

Can hear an axis-symmetric drum!

Call Ω *dynamically spectrally rigid* (DSR) if any smooth isopectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$ is an isometry, i.e. $\mathcal{L}(\Omega_t) \equiv \mathcal{L}(\Omega_0)$.

Theorem

De Simoi-K-Wei An axis-symmetric domain near the circle is DSR.

Can hear an axis-symmetric drum!

Call Ω *dynamically spectrally rigid* (DSR) if any smooth isopectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$ is an isometry, i.e. $\mathcal{L}(\Omega_t) \equiv \mathcal{L}(\Omega_0)$.

Theorem

De Simoi-K-Wei *An axis-symmetric domain near the circle is DSR.*

Can hear an axis-symmetric drum!

Call Ω *dynamically spectrally rigid* (DSR) if any smooth isopectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$ is an isometry, i.e. $\mathcal{L}(\Omega_t) \equiv \mathcal{L}(\Omega_0)$.

Theorem

De Simoi-K-Wei *An axis-symmetric domain near the circle is DSR.*

Can hear an axis-symmetric drum!

Call Ω *dynamically spectrally rigid* (DSR) if any smooth isopectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$ is an isometry, i.e. $\mathcal{L}(\Omega_t) \equiv \mathcal{L}(\Omega_0)$.

Theorem

De Simoi-K-Wei *An axis-symmetric domain near the circle is DSR.*

Can hear an axis-symmetric drum!

Call Ω *dynamically spectrally rigid* (DSR) if any smooth isopectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$ is an isometry, i.e. $\mathcal{L}(\Omega_t) \equiv \mathcal{L}(\Omega_0)$.

Theorem

De Simoi-K-Wei *An axis-symmetric domain near the circle is DSR.*

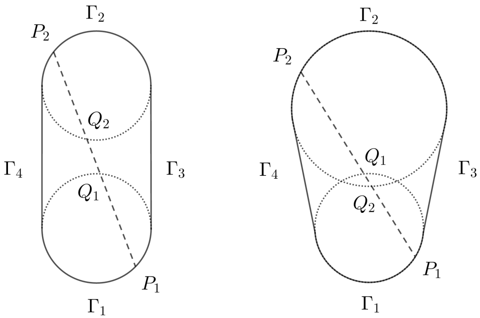
Can hear an axis-symmetric drum!

Call Ω *dynamically spectrally rigid* (DSR) if any smooth isopectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$ is an isometry, i.e. $\mathcal{L}(\Omega_t) \equiv \mathcal{L}(\Omega_0)$.

Theorem

De Simoi-K-Wei *An axis-symmetric domain near the circle is DSR.*

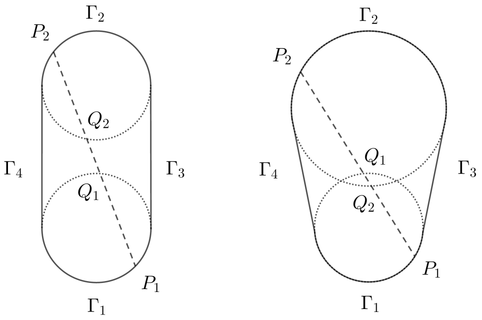
Can't deform isospectrally a peicewise analytic Bunimovich drum!



Theorem

J. Chen-K-H. Zhang A p.a. Bunimovich stadium is DSR. In addition, a p.a. Bunimovich squash-like stadium is DSR.

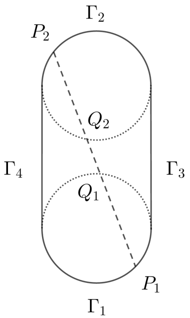
Can't deform isospectrally a peicewise analytic Bunimovich drum!

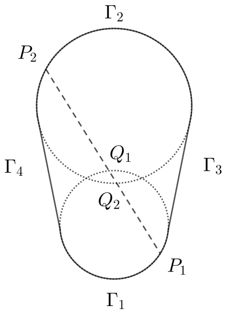


Theorem

J. Chen-K-H. Zhang A p.a. Bunimovich stadium is DSR. In addition, a p.a. Bunimovich squash-like stadium is DSR.

Can't deform isospectrally a peicewise analytic Bunimovich drum!

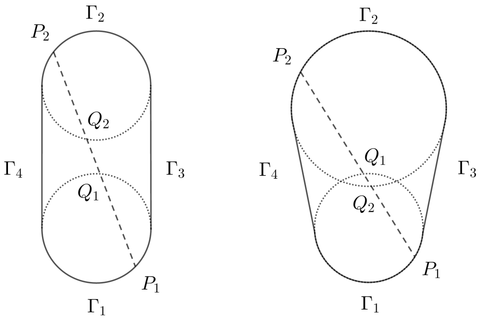




Theorem

J. Chen-K-H. Zhang A p.a. Bunimovich stadium is DSR. In addition, a p.a. Bunimovich squash-like stadium is DSR.

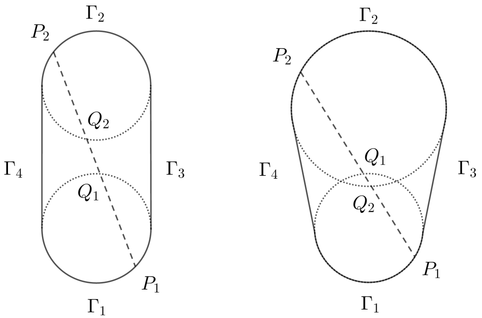
Can't deform isospectrally a peicewise analytic Bunimovich drum!



Theorem

J. Chen-K-H. Zhang A p.a. Bunimovich stadium is DSR. In addition, a p.a. Bunimovich squash-like stadium is DSR.

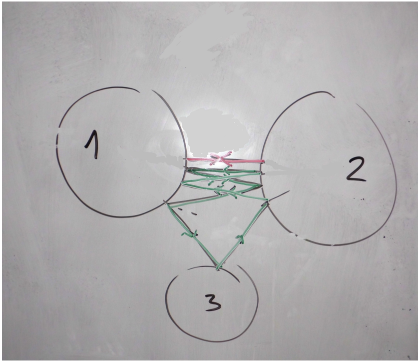
Can't deform isospectrally a peicewise analytic Bunimovich drum!



Theorem

J. Chen-K-H. Zhang A p.a. Bunimovich stadium is DSR. In addition, a p.a. Bunimovich squash-like stadium is DSR.

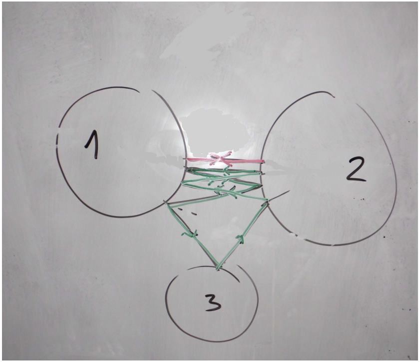
Dispercing (hyperbolic) billiard



Theorem

De Simoi-K-Leguil Marked Length Spectrum determins an analytic three disk system with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries.

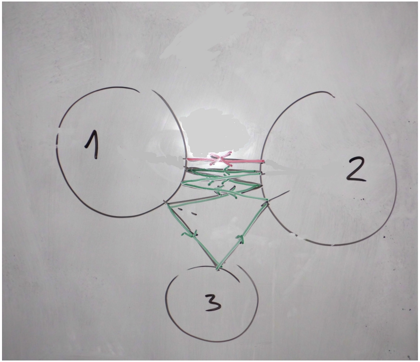
Dispercing (hyperbolic) billiard



Theorem

De Simoi-K-Leguil Marked Length Spectrum determins an analytic three disk system with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries.

Dispercing (hyperbolic) billiard



Theorem

De Simoi-K-Leguil Marked Length Spectrum determins an analytic three disk system with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries.

Marked Length spectrum, Spectral & Global Rigidity

Let (S, g) be a negatively curved compact surface.

Call the union of minimal geodesics in each homotopy class γ

$$\mathcal{L}(S, g) = \cup(\ell_\gamma, \gamma)$$

the marked length spectrum.

Guillemin-Kazhdan'80 any (S, g) is spectrally rigid.

Croke, Otal'90 the marked length spectrum determines (S, g) upto isometry.

Croke-Sharafutdinov'98 any negatively curved manifold (M, g) is spectrally rigid.

Marked Length spectrum, Spectral & Global Rigidity

Let (S, g) be a negatively curved compact surface.

Call the union of minimal geodesics in each homotopy class γ

$$\mathcal{L}(S, g) = \cup(\ell_\gamma, \gamma)$$

the marked length spectrum.

Guillemin-Kazhdan'80 any (S, g) is spectrally rigid.

Croke, Otal'90 the marked length spectrum determines (S, g) upto isometry.

Croke-Sharafutdinov'98 any negatively curved manifold (M, g) is spectrally rigid.

Marked Length spectrum, Spectral & Global Rigidity

Let (S, g) be a negatively curved compact surface.

Call the union of minimal geodesics in each homotopy class γ

$$\mathcal{L}(S, g) = \cup(\ell_\gamma, \gamma)$$

the marked length spectrum.

Guillemin-Kazhdan'80 any (S, g) is spectrally rigid.

Croke, Otal'90 the marked length spectrum determines (S, g) upto isometry.

Croke-Sharafutdinov'98 any negatively curved manifold (M, g) is spectrally rigid.

Marked Length spectrum, Spectral & Global Rigidity

Let (S, g) be a negatively curved compact surface.

Call the union of minimal geodesics in each homotopy class γ

$$\mathcal{L}(S, g) = \cup(\ell_\gamma, \gamma)$$

the marked length spectrum.

Guillemin-Kazhdan'80 any (S, g) is spectrally rigid.

Croke, Otal'90 the marked length spectrum determines (S, g) upto isometry.

Croke-Sharafutdinov'98 any negatively curved manifold (M, g) is spectrally rigid.

Marked Length spectrum, Spectral & Global Rigidity

Let (S, g) be a negatively curved compact surface.

Call the union of minimal geodesics in each homotopy class γ

$$\mathcal{L}(S, g) = \cup(\ell_\gamma, \gamma)$$

the marked length spectrum.

Guillemin-Kazhdan'80 any (S, g) is spectrally rigid.

Croke, Otal'90 the marked length spectrum determines (S, g) upto isometry.

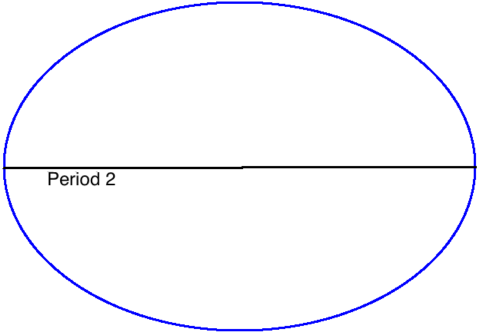
Croke-Sharafutdinov'98 any negatively curved manifold (M, g) is spectrally rigid.

Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics. Birkhoff proved

Lemma

For any convex domain Ω and any $q > 1$ there is a periodic orbit of period q, given by inscribed q-gons and denoted $S_q = S_q(\Omega)$. If Ω is axis-symmetric, then S_q can be chosen axis-symmetric.

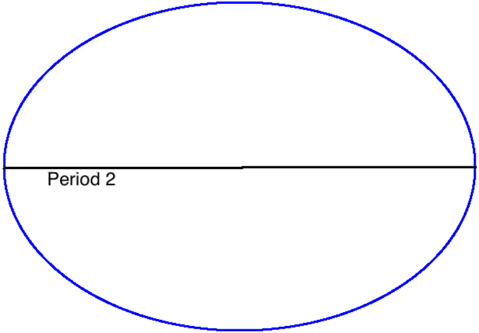


Ideas of proof of Dynamical Spectral Rigidity

- ‘Skeleton’ of the dynamics. Birkhoff proved

Lemma

For any convex domain Ω and any $q > 1$ there is a periodic orbit of period q, given by inscribed q-gons and denoted $S_q = S_q(\Omega)$. If Ω is axis-symmetric, then S_q can be chosen axis-symmetric.

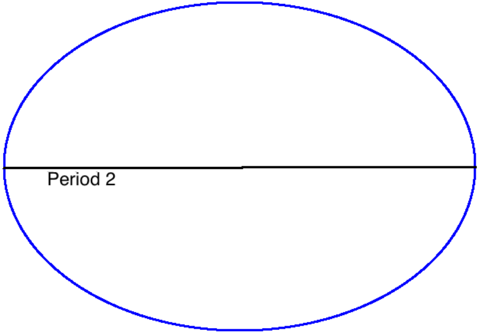


Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics. Birkhoff proved

Lemma

For any convex domain Ω and any $q > 1$ there is a periodic orbit of period q, given by inscribed q-gons and denoted $S_q = S_q(\Omega)$. If Ω is axis-symmetric, then S_q can be chosen axis-symmetric.

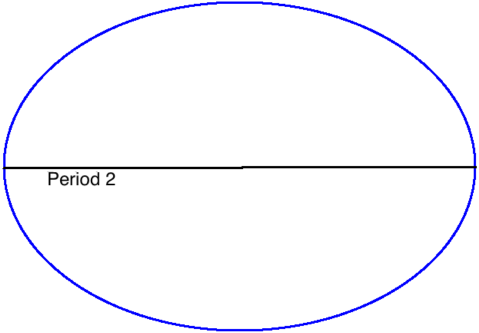


Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics. Birkhoff proved

Lemma

For any convex domain Ω and any $q > 1$ there is a periodic orbit of period q, given by inscribed q-gons and denoted $S_q = S_q(\Omega)$. If Ω is axis-symmetric, then S_q can be chosen axis-symmetric.

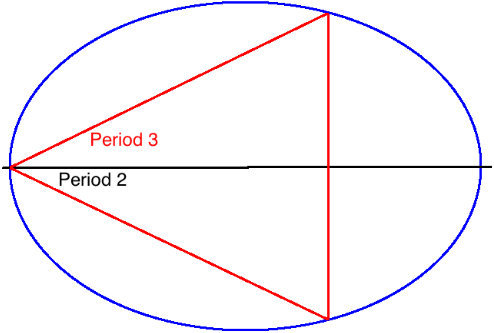


Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics. Birkhoff proved

Lemma

For any convex domain Ω and any $q > 1$ there is a periodic orbit of period q, given by inscribed q-gons and denoted $S_q = S_q(\Omega)$. If Ω is axis-symmetric, then S_q can be chosen axis-symmetric.

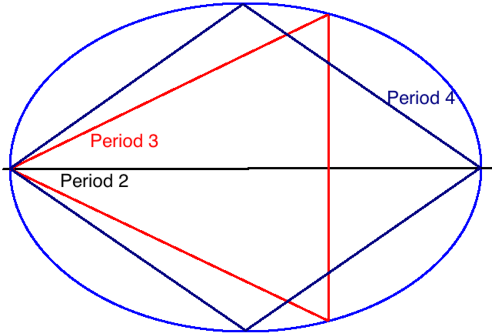


Ideas of proof of Dynamical Spectral Rigidity

- ‘Skeleton’ of the dynamics. Birkhoff proved

Lemma

For any convex domain Ω and any $q > 1$ there is a periodic orbit of period q, given by inscribed q-gons and denoted $S_q = S_q(\Omega)$. If Ω is axis-symmetric, then S_q can be chosen axis-symmetric.

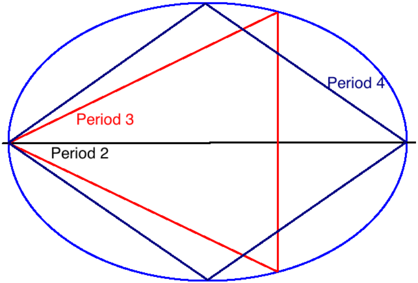


Ideas of proof of Dynamical Spectral Rigidity

- ‘Skeleton’ of the dynamics: symmetric q-gons
  $S_q = (x_q^{(k)}, \varphi_q^{(k)}),\ q > 1.$ $S^r(\mathbb{T})$ – space of C^r-symmetric functions.
- Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$,

$$\partial\Omega_t = \partial\Omega_0 + t \cdot n(s) + O(t^2), \qquad n \in S^r(\mathbb{T}).$$

  Then $\ell_q(n) = \sum_{k=1}^{q} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0.$

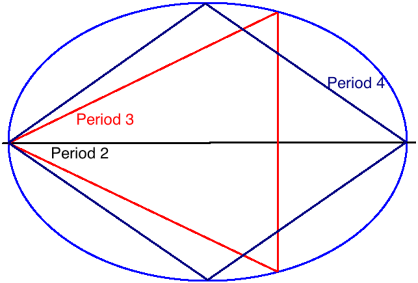


Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics: symmetric q-gons
  $S_q = (x_q^{(k)}, \varphi_q^{(k)}),\ q > 1$. $S^r(\mathbb{T})$ – space of C^r-symmetric functions.
- Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$,

$$\partial\Omega_t = \partial\Omega_0 + t \cdot n(s) + O(t^2), \qquad n \in S^r(\mathbb{T}).$$

Then $\ell_q(n) = \sum_{k=1}^{q} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0$.

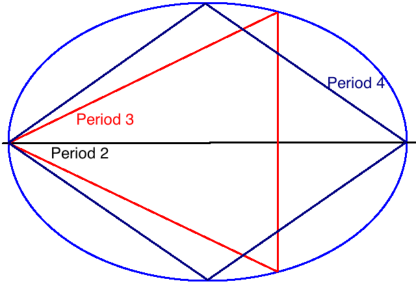


Ideas of proof of Dynamical Spectral Rigidity

- ‘Skeleton’ of the dynamics: symmetric q-gons
  $S_q = (x_q^{(k)}, \varphi_q^{(k)}),\ q > 1$. $S^r(\mathbb{T})$ – space of C^r-symmetric functions.
- Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$,

$$\partial\Omega_t = \partial\Omega_0 + t \cdot n(s) + O(t^2), \qquad n \in S^r(\mathbb{T}).$$

Then $\ell_q(n) = \sum_{k=1}^{q} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0$.

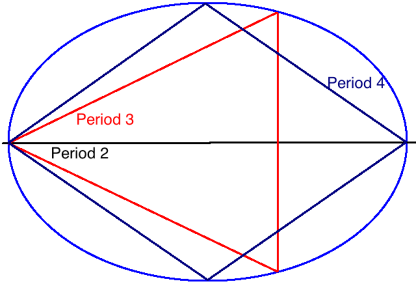


Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics: symmetric q-gons
  $S_q = (x_q^{(k)}, \varphi_q^{(k)}),\ q > 1$.$S^r(\mathbb{T})$ – space of C^r-symmetric functions.
- Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$,

$$\partial\Omega_t = \partial\Omega_0 + t \cdot n(s) + O(t^2), \qquad n \in S^r(\mathbb{T}).$$

Then $\ell_q(n) = \sum_{k=1}^{q} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0.$



Ideas of proof of Dynamical Spectral Rigidity

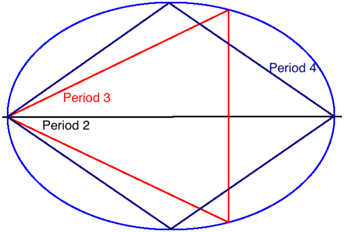
- 'Skeleton' of the dynamics: symmetric q-gons $S_q = (x_q^{(k)}, \varphi_q^{(k)}),\ q > 1$. $S^r(\mathbb{T})$ – space of C^r-symmetric functions.
- Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$,

$$\partial\Omega_t = \partial\Omega_0 + t \cdot n(s) + O(t^2), \qquad n \in S^r(\mathbb{T}).$$

Then $\quad \ell_q(n) = \sum_{k=1}^{q} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0.$

- Define a linearized isospectral operator

$$\mathcal{L}_\Omega : C^r(\mathbb{T}) \to \ell^\infty, \qquad \mathcal{L}_\Omega(n) = (\ell_q(n),\ q = 0, 1, \dots).$$

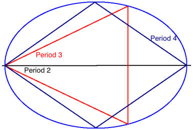


Ideas of proof of Dynamical Spectral Rigidity

- 'Skeleton' of the dynamics: symmetric q-gons
  $S_q = (x_q^{(k)}, \varphi_q^{(k)}),\ q > 1$. $S^r(\mathbb{T})$ – space of C^r-symmetric functions.
- Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$,
  $$\partial\Omega_t = \partial\Omega_0 + t \cdot n(s) + O(t^2), \qquad n \in S^r(\mathbb{T}).$$
  Then $\quad \ell_q(n) = \sum_{k=1}^{q} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0.$
- Define a linearized isospectral operator
  $$\mathcal{L}_\Omega : C^r(\mathbb{T}) \to \ell^\infty, \qquad \mathcal{L}_\Omega(n) = (\ell_q(n),\ q = 0, 1, \dots).$$

Lemma

If $\mathcal{L}_\Omega$ is injective, then Ω is DSR.



The Linearized Isospectral Operator for the circle

Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$, of the circle. In polar coordinates $(r, s) \in \mathbb{R}_+ \times \mathbb{T}$

$$\partial\Omega_t = \{r = 1 + t \cdot n(s) + O(t^2)\}, \qquad n \in S^r(\mathbb{T}).$$

Then

$$\ell_q(n) = \sum_{k=1}^{q} n\left(\frac{k}{q}\right) = 0.$$

Lemma

Let $n(s) = \sum_{k \in \mathbb{Z}_+} n_k \cos ks$ be the Fourier expansion. Then $\ell_q(n) = 0$ implies $n_{kq} = 0$ for $k \geq 1$.

Lemma

The Linearized Isospectral Operator $\mathcal{L}_{\Omega_0}$ is injective and is an upper triangular with units on the diagonal.

The Linearized Isospectral Operator for the circle

Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$, of the circle. In polar coordinates $(r, s) \in \mathbb{R}_+ \times \mathbb{T}$

$$\partial\Omega_t = \{r = 1 + t \cdot n(s) + O(t^2)\}, \qquad n \in S^r(\mathbb{T}).$$

Then

$$\ell_q(n) = \sum_{k=1}^{q} n(\frac{k}{q}) = 0.$$

Lemma

Let $n(s) = \sum_{k\in\mathbb{Z}_+} n_k \cos ks$ be the Fourier expansion. Then $\ell_q(n) = 0$ implies $n_{kq} = 0$ for $k \geq 1$.

Lemma

The Linearized Isospectral Operator $\mathcal{L}_{\Omega_0}$ is injective and is an upper triangular with units on the diagonal.

The Linearized Isospectral Operator for the circle

Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$, of the circle. In polar coordinates $(r, s) \in \mathbb{R}_+ \times \mathbb{T}$

$$\partial\Omega_t = \{r = 1 + t \cdot n(s) + O(t^2)\}, \qquad n \in S^r(\mathbb{T}).$$

Then

$$\ell_q(n) = \sum_{k=1}^{q} n(\frac{k}{q}) = 0.$$

Lemma

Let $n(s) = \sum_{k\in\mathbb{Z}_+} n_k \cos ks$ be the Fourier expansion. Then $\ell_q(n) = 0$ implies $n_{kq} = 0$ for $k \geq 1$.

Lemma

The Linearized Isospectral Operator $\mathcal{L}_{\Omega_0}$ is injective and is an upper triangular with units on the diagonal.

The Linearized Isospectral Operator for the circle

Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$, of the circle. In polar coordinates $(r, s) \in \mathbb{R}_+ \times \mathbb{T}$

$$\partial\Omega_t = \{r = 1 + t \cdot n(s) + O(t^2)\}, \qquad n \in S^r(\mathbb{T}).$$

Then

$$\ell_q(n) = \sum_{k=1}^{q} n(\frac{k}{q}) = 0.$$

Lemma

Let $n(s) = \sum_{k \in \mathbb{Z}_+} n_k \cos ks$ be the Fourier expansion. Then $\ell_q(n) = 0$ implies $n_{kq} = 0$ for $k \geq 1$.

Lemma

The Linearized Isospectral Operator $\mathcal{L}_{\Omega_0}$ is injective and is an upper triangular with units on the diagonal.

The Linearized Isospectral Operator for the circle

Consider an isospectral deformation $\{\Omega_t\}_t \subset \mathcal{S}^r$, of the circle. In polar coordinates $(r, s) \in \mathbb{R}_+ \times \mathbb{T}$

$$\partial\Omega_t = \{r = 1 + t \cdot n(s) + O(t^2)\}, \qquad n \in S^r(\mathbb{T}).$$

Then

$$\ell_q(n) = \sum_{k=1}^{q} n(\frac{k}{q}) = 0.$$

Lemma

Let $n(s) = \sum_{k \in \mathbb{Z}_+} n_k \cos ks$ be the Fourier expansion. Then $\ell_q(n) = 0$ implies $n_{kq} = 0$ for $k \geq 1$.

Lemma

The Linearized Isospectral Operator $\mathcal{L}_{\Omega_0}$ is injective and is an upper triangular with units on the diagonal.

The X-ray transform

Let g_0 be a negatively curved metric. Let g be close to g_0. Denote $h = g_0 - g$ and γ_c the closed geodesic in the homotopy class c. Consider the first variation:

$$(d\mathcal{L}(g_0) \cdot h)(c) = \frac{1}{2L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\dot{\gamma}_c(t), \dot{\gamma}_c(t))dt.$$

The X-ray transform and solenoidal injectivity

Let g_0 be a negatively curved metric. Let g be close to g_0. Denote $C^\infty(M, S^2T^*M)$ the space of symmetric 2-tensors, $h = g_0 - g$ be the symmetric 2-tensor and γ_c the closed geodesic in the homotopy class c. Call

$$(I_{g_0} \cdot h)(c) = \frac{1}{2L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\dot\gamma_c(t), \dot\gamma_c(t))dt.$$

the X-transform. The divergence operator is its formal adjoint given by $D^*f := -Tr(\nabla f)$ with the trace Tr naturally defined.

Theorem

Paternain-Salo-Uhlmann *Let (M, g_0) be a smooth Riemannian manifold and assume that the geodesic flow of g_0 is Anosov. Then I_{g_0} is solenoidal injective, i.e.*

$$\ker I_{g_0} \cap C^\infty(M, S^2T^*M) \cap \ker D^* = 0.$$

The X-ray transform and solenoidal injectivity

Let g_0 be a negatively curved metric. Let g be close to g_0. Denote $C^\infty(M, S^2T^*M)$ the space of symmetric 2-tensors, $h = g_0 - g$ be the symmetric 2-tensor and γ_c the closed geodesic in the homotopy class c. Call

$$(I_{g_0} \cdot h)(c) = \frac{1}{2L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\dot\gamma_c(t), \dot\gamma_c(t))dt.$$

the X-transform. The divergence operator is its formal adjoint given by $D^*f := -Tr(\nabla f)$ with the trace Tr naturally defined.

Theorem

Paternain-Salo-Uhlmann *Let (M, g_0) be a smooth Riemannian manifold and assume that the geodesic flow of g_0 is Anosov. Then I_{g_0} is solenoidal injective, i.e.*

$$\ker I_{g_0} \cap C^\infty(M, S^2T^*M) \cap \ker D^* = 0.$$