Bernoulli and K properties in smooth dynamics

Adam Kanigowski

08.14.2019
2020 Vision for Dynamics

Bernoulli shifts

- $\Sigma = \{1, \ldots, d\}^{\mathbb{Z}}$;
- $\mathbf{p} = (p_1, \ldots, p_d)$, $\sum_{i=1}^{d} p_i = 1$ – probability vector;
- $\sigma : (\Sigma, \mathbf{p}^{\mathbb{Z}}) \to (\Sigma, \mathbf{p}^{\mathbb{Z}})$ – Bernoulli shift,

$$\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}.$$

Bernoulli systems

$T \in Aut(X, \mathcal{B}, \mu)$ is a Bernoulli system (or Bernoulli), if T is isomorphic to a Bernoulli shift.

K-systems

$T \in Aut(X, \mathcal{B}, \mu)$ is a K-system if every (non-trivial) factor of T has positive entropy.

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Bernoulli and K properties

- T is Bernoulli $\rightarrow$ T is K (Ornstein, 1970);
- K does NOT imply Bernoulli (Ornstein, 1975);
- $(\sigma, \sigma^{-1})(x, y) = (\sigma(x), \sigma^{(-1)^{x_0}}(y))$ is K and NOT Bernoulli (Kalikow, 1980).

General problem

Bernoulli and K properties in smooth dynamics.

Smooth setting

- M – compact, connected, smooth manifold;
- μ – smooth density on M;
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- Existence of Bernoulli systems (topological obstructions);
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Existence of smooth Bernoulli systems

$dim\ M = 1$

NOT possible – Denjoy theory.

Katok, 1979

There exists a smooth Bernoulli system T on every surface ($dim\ M = 2$). In fact, T has non-zero exponents and hence in Bernoulli by Pesin theory.

Brin, Feldman, Katok, 1981

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Smooth K non Bernoulli systems

Skew products

$A \in Diff^{\infty}(M)$ hyperbolic, (K_t) smooth flow on N, $\varphi : M \to \mathbb{R}$ smooth

$$T(x, y) = (Ax, K_{\varphi(x)}y).$$

Smooth K non Bernoulli systems

- smooth skew-products in dimension 8 – (Katok, 1980) $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\varphi : \mathbb{T}^2 \to \mathbb{R}$ smooth non-coboundary, $K_t = h_t \times h_t$, where (h_t) is the horocycle flow.
- smooth version of (σ, σ^{-1}) in dimension 5 – (Rudolph, 1988) $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\varphi : \mathbb{T}^2 \to \mathbb{R}$ smooth non-coboundary with zero mean, $K_t = g_t$, (g_t) the geodesic flow;

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Translations on semisimple Lie groups

G – semisimple Lie group; Γ – irreducible lattice in G, $g \in G$,
$L_g(x\Gamma) = (gx)\Gamma$ on $(G/\Gamma, \mu_{Haar})$.

K-property, Conze, 1972; Dani 1974.

If L_g has positive entropy then L_g is K.

Bernoulli property, Dani 1977

If L_g has positive entropy and Ad_g is diagonalizable over $\mathbb{C}$ on the center space then L_g is Bernoulli.

Example

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}$$ is NOT diagonalizable over $\mathbb{C}$.

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$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}$$ is NOT diagonalizable over $\mathbb{C}$.

Translations on semisimple Lie groups

G – semisimple Lie group; Γ – irreducible lattice in G, $g \in G$,
$L_g(x\Gamma) = (gx)\Gamma$ on $(G/\Gamma, \mu_{Haar})$.

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Dani's proof of Bernoulli property uses the fact that the action of L_g on the center space is isometric. This is crucial to apply the Ornstein-Weiss machinery.

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More general setting

L_g is an example of a system with the following properties:

Properties of L_g:

a. partial hyperbolicity and dynamical coherence;

b. zero exponents in the center space;

c. exponential mixing, i.e. for ϕ, ψ sufficiently smooth

$$\left| \int_{G/\Gamma} \phi \cdot (\psi \circ L_g^n) d\mu_{Haar} \right| \leq \|\phi\|_k \|\psi\|_k e^{-\eta n}.$$

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If $f \in Diff^\infty(M, \mu)$ satisfies a.,b.,c., then f IS Bernoulli.

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Main Idea

Ornstein-Weiss reduction

Let $W^u(x,\delta)$ and $W^u(y,\delta)$ be nearby unstable leaves of size δ. If for every N there exists an almost measure preserving map $\theta_{x,y,\delta,N} : (W^u(x,\delta), m^u_x) \to (W^u(y,\delta), m^u_y)$ such that

$$T^n z \text{ and } T^n\theta z \text{ are close for most } 0 \leq n \leq N.$$

then T is Bernoulli.

Examples

- if T is hyperbolic, then θ is the stable holonomy.
- it T is isometric on the center space, then θ is the center-stable holonomy.
- if T in NOT isometric on the center space, then there is no obvious choice for θ.

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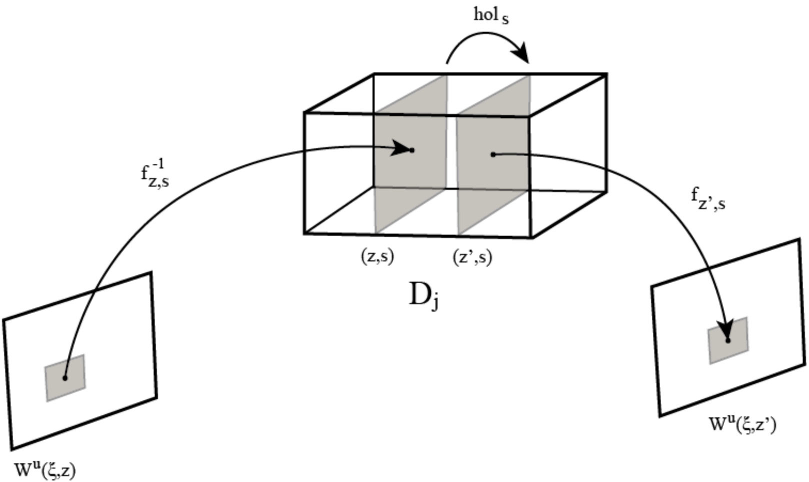
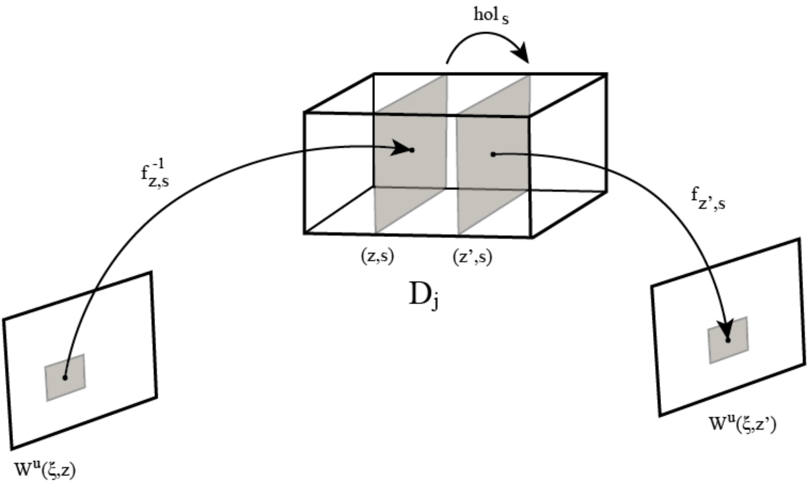


PROBLEM: how do we know this is measure preserving?



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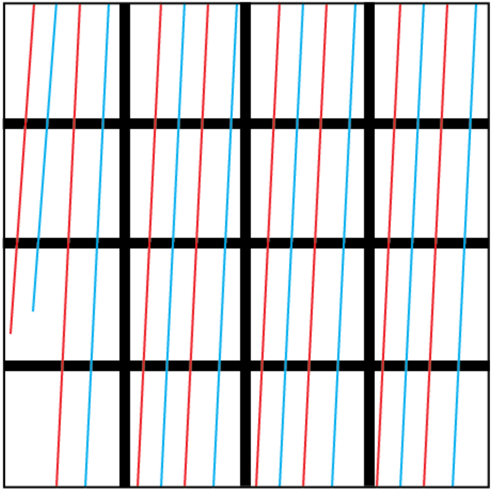

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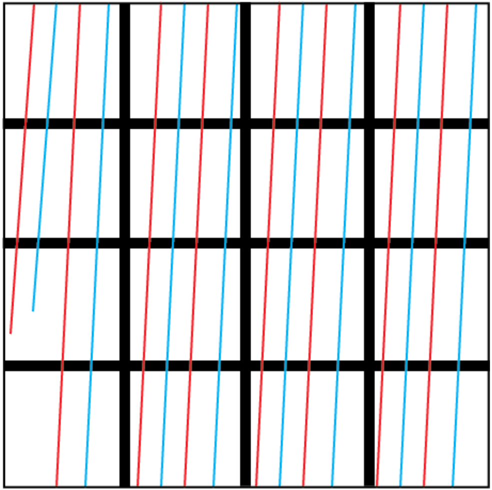

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Consequences

- Exponential mixing implies positive entropy;
- If an algebraic action is exponentially mixing then it is Bernoulli;
- K and NOT Bernoulli systems don't have exponentially mixing (nice) smooth realizations;
- Skew product non-Bernoulli transformations, i.e.

$$T(x, y) = (\sigma x, S_{\varphi(x)} y)$$

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Question 1

Does K imply Bernoulli in dimension 3?

Question 2

Let (h_t) be a horocycle flow and let
$T : \{0,1\}^{\mathbb{Z}} \times SL(2,\mathbb{R})/\Gamma \to \{0,1\}^{\mathbb{Z}} \times SL(2,\mathbb{R})/\Gamma$,

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Then T is a K system (Katok, 1980). Is T Bernoulli?

Question 3

Is there a relation between growth on the center and mixing properties that would imply Bernoulli?

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THANK YOU!