Furstenberg theorem; now with a parameter!

Joint work with A. Gorodetski

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After n iterations we have

$$F_{n,\omega} = f_{\omega_n} \circ \dots \circ f_{\omega_1}.$$

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The really simplest case:

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Theorem (Furstenberg)

If there is no common invariant measure, nor a finite invariant union of subspaces, then $\lambda_{FK} > 0$.

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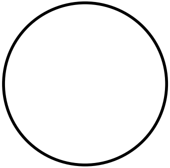

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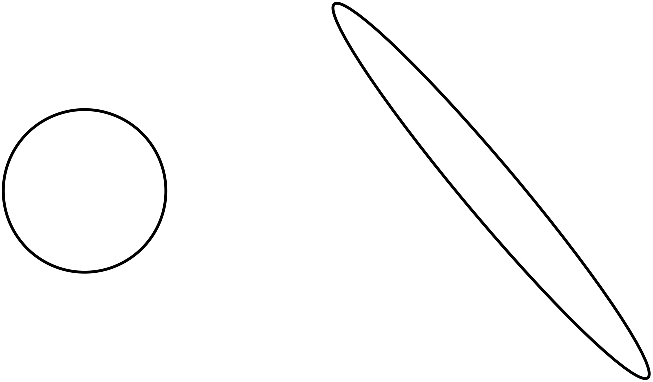

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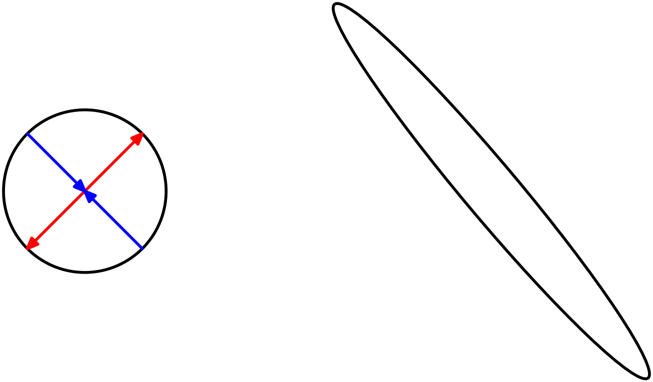

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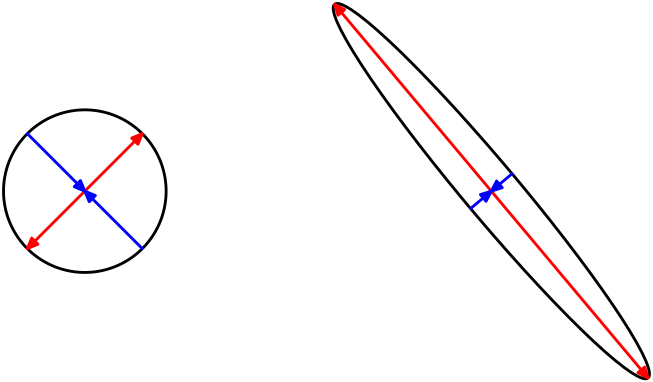

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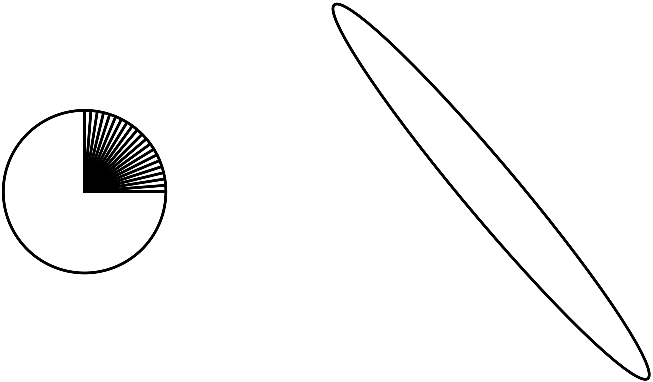

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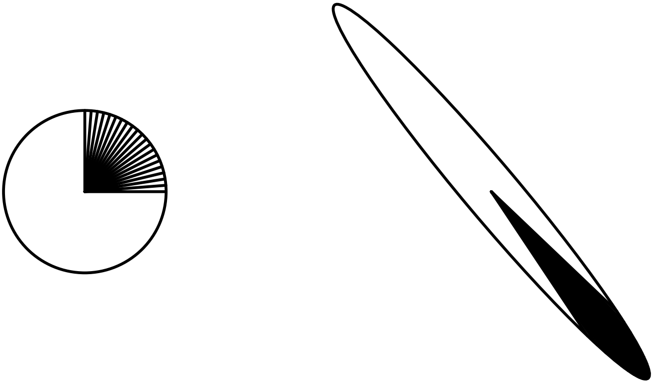

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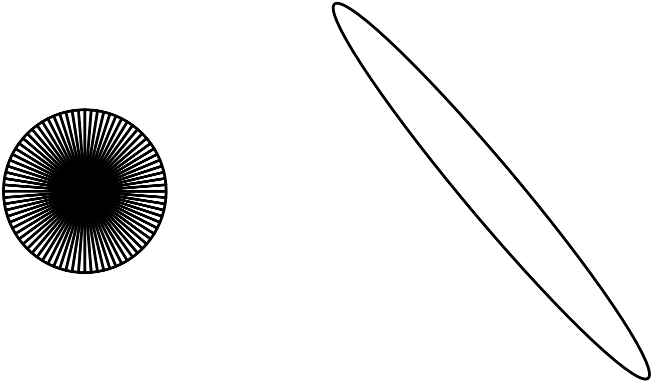

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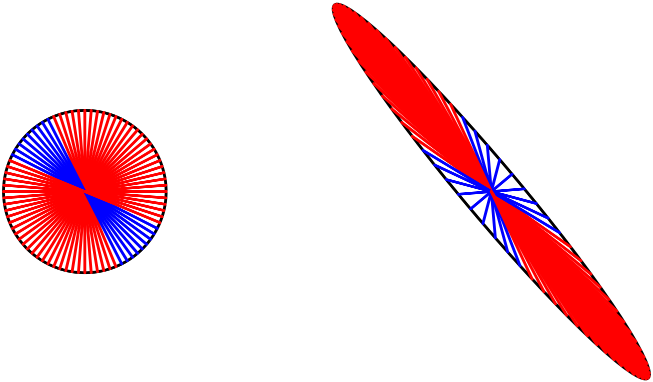

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But what if we first fix ω and then vary $a \in J$?

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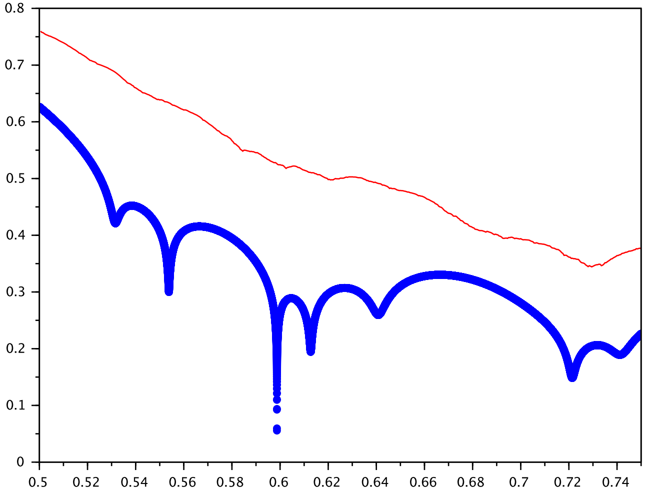

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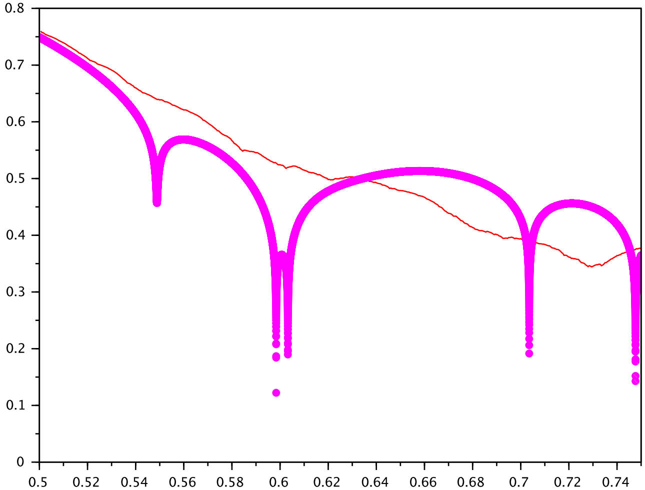
Another product of length 30:



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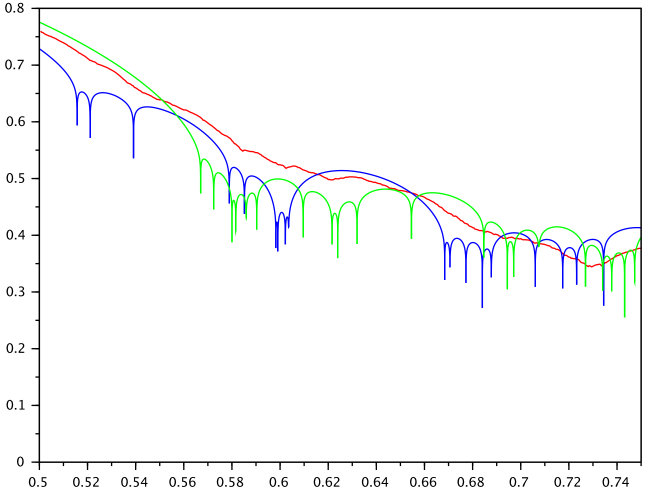
Two products of length 100:



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Under some assumptions (see below), almost surely:

- *For any $a \in J$,*

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$$\forall a \in X_0 \quad \liminf_{n\to\infty} \frac{1}{n} \log \| T_{n,a,\bar{\omega}} \| = 0.$$

- *The (random) set of parameters with exceptional behaviour,*

$$X_{ex} := \left\{ a \in J \mid \liminf_{n\to\infty} \frac{1}{n} \log \| T_{n,a,\bar{\omega}} \| < \lambda_{FK}(a) \right\},$$

  has zero Hausdorff dimension.

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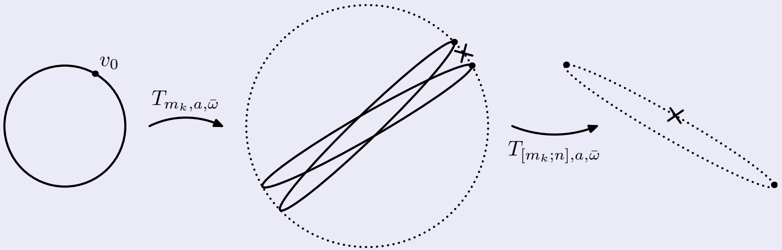

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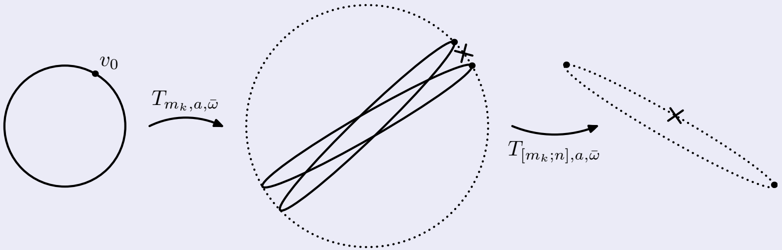

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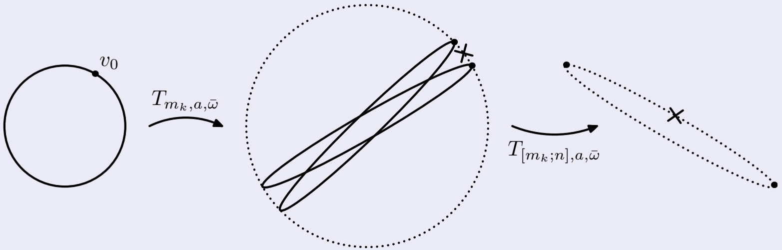
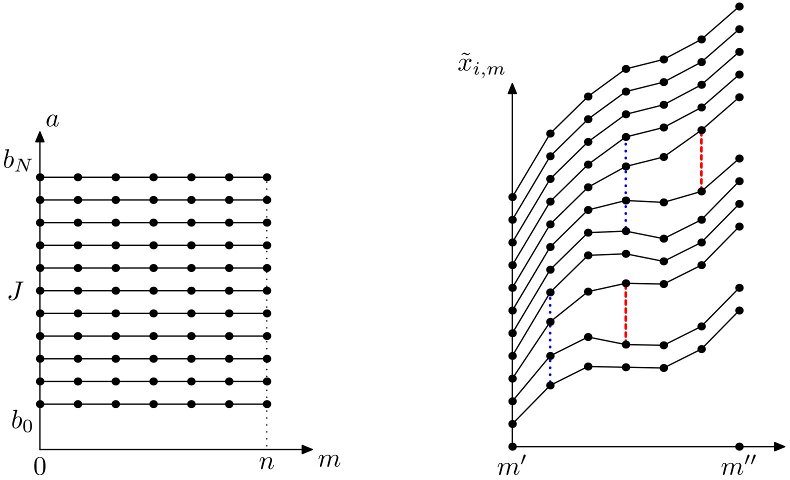


Idea of the proof



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What if the potential $V(x)$ is random?
Discrete case: $\psi \in l_2(\mathbb{Z})$; then $\Delta\psi$ is replaced with

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Motivation: one-dimensional Anderson localization

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An electron in one-dimensional crystal (changing sign, V and E):

$$\widehat{H}[\psi](n) = \psi(n+1) + \psi(n-1) + \widetilde{V}(n)\psi(n)$$

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For random i.i.d. $V(n)$'s, $V(n) = \omega_n$, we have a product

$$T_{\omega,n;E} = A_{\omega_n,E} \dots A_{\omega_1,E}.$$

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- But for a general E, the vectors v_- and v_+ are different! (Well, it couldn't be otherwise: otherwise we would have too many eigenfunctions!)
- The (countably many) parameter values at which these vectors coincide can be found by considering finite products from $-N$ to N and then increasing N.