Loosely Bernoulli odometer-based systems whose corresponding circular systems are not loosely Bernoulli

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The Isomorphism Problem

Important question dating back to the foundational paper of von Neumann (1932):

ZUR OPERATORENMETHODE IN DER KLASSISCHEN MECHANIK[1].

VON J. v. NEUMANN, PRINCETON.

The Isomorphism Problem

Classify ergodic transformations up to measure isomorphism.

classification is a method of determining isomorphism between transformations, perhaps by computing other invariants for which equivalence is easy to determine.

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Determine when two transformations are isomorphic.

- Halmos-von Neumann (1942): The spectrum of the associated Koopman operator ($U_T : L^2(X) \to L^2(X)$, $U_T f = f \circ T$) is a complete isomorphism invariant for ergodic transformations with pure point spectrum

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  BUT: Construction of uncountable family of non-isomorphic K-automorphisms with the same entropy (Ornstein-Shields 1973)

von Neumann's Classification problem is impossible

Theorem (Foreman-Rudolph-Weiss 2011)

$$\{(S,T) \mid S \text{ and } T \text{ are ergodic and isomorphic}\} \subseteq \mathcal{E}(X) \times \mathcal{E}(X)$$

is a complete analytic set. In particular, it is not Borel.

Descriptively: determining isomorphism between ergodic transformations is inaccessible to countable methods that use countable amount of information.

M. Foreman, D. Rudolph, B. Weiss
The conjugacy problem in ergodic theory
Annals of Mathematics 173 (2011): 1529-1586

Basics of Descriptive Set Theory

Let X, Y be Polish spaces (i.e. separable completely metrizable topol. spaces).

Definition: Reduction

Let $A \subseteq X$ and $B \subseteq Y$. A function $f : X \to Y$ **reduces** A to B iff for all $x \in X$:

$$x \in A \text{ if and only if } f(x) \in B.$$

Such f is called a Borel (resp. continuous) reduction if f is a Borel (resp. continuous) function.

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Definition: Complete analytic set

An analytic set A is called **complete analytic** iff every analytic set can be reduced to A by a Borel function.

Since there are analytic non-Borel sets, a complete analytic set is not Borel.

Canonical example of a complete analytic set

$\mathbb{N}^{<\mathbb{N}}$: finite sequences of natural numbers
A **tree** is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that if $\tau \in T$ and σ is an initial segment of τ, then $\sigma \in T$.
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An **infinite branch** through a tree T is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$(f(0), f(1), \dots, f(n-1)) \in T.$$

A tree is called **ill-founded** iff it has an infinite branch.

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Classical fact

The collection of ill-founded trees is a complete analytic subset of $\mathcal{T}REES$.

Idea of proof

Construct a continuous function $F : \mathcal{TREES} \to \mathcal{E}(X)$ such that for $\mathcal{T} \in \mathcal{TREES}$:

$$\mathcal{T} \text{ has an infinite branch if and only if } F(\mathcal{T}) \cong F(\mathcal{T})^{-1}.$$

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Use

$$i : \mathcal{E}(X) \to \mathcal{E}(X) \times \mathcal{E}(X),\; i(T) = (T, T^{-1})$$

to reduce $\left\{ T \;\middle|\; T \cong T^{-1} \right\}$ to $\{(S, T) \mid S \cong T\}$.
Hence, $\{(S, T) \mid S \cong T\}$ is complete analytic.

Symbolic systems

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Let Σ be a finite alphabet.

Definition: Construction sequence

A *construction sequence* is a sequence of collections of words $(W_n)_{n\in\mathbb{N}}$, satisfying the following properties:

1. for every $n \in \mathbb{N}$ all words in W_n have the same length h_n,
2. each $w \in W_n$ occurs at least once as a subword of each $w' \in W_{n+1}$,
3. there is a summable sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers such that for every $n \in \mathbb{N}$, every word $w \in W_{n+1}$ can be uniquely parsed into segments $u_0 w_1 u_1 w_1 \dots w_l u_{l+1}$ such that each $w_i \in W_n$, each u_i (called spacer or boundary) is a word in Σ of finite length and for this parsing

$$\frac{\sum_{i=0}^{l+1} |u_i|}{h_{n+1}} < \varepsilon_{n+1}.$$

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Let $\mathbb{K}$ be the collection of $x \in \Sigma^{\mathbb{Z}}$ such that every finite contiguous substring of x occurs inside some $w \in W_n$.

Odometer-based systems

Definition: Unique readability

Let Σ be a language and W be a collection of finite words in Σ. Then W is *uniquely readable* iff whenever $u, v, w \in W$ and $uv = pws$ with p and s strings of symbols in Σ, then either p or s is the empty word.

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Definition: Uniformity

We call a construction sequence $(W_n)_{n\in\mathbb{N}}$ *uniform* if for each $n \in \mathbb{N}$ there is a constant $c > 0$ such that for all words $w' \in W_{n+1}$ and $w \in W_n$ the number of occurrences of w in w' is equal to c.

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Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers $k_n \geq 2$.

Definition: Odometer-based systems

Let $(\mathtt{W}_n)_{n\in\mathbb{N}}$ be a uniquely readable construction sequence with $\mathtt{W}_0 = \Sigma$ and $\mathtt{W}_{n+1} \subseteq (\mathtt{W}_n)^{k_n}$ for every $n \in \mathbb{N}$. The associated symbolic shift will be called an *odometer-based system*.

Smooth Ergodic Theory

Another important question dating back to the foundational paper of von Neumann (1932):

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590 J. V. NEUMANN.

morphieinvarianten Eigenschaften. Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden[13], vielleicht sogar eine stetig-differentiierbare, oder gar eine mechanische. Dies mag es rechtfertigen, daß hier an Stelle der eigentlich interessanten mechanischen Strömungen alle allgemeinen untersucht werden.

[13] Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben.

Smooth realization problem

FIVE MOST RESISTANT PROBLEMS
IN DYNAMICS

A. Katok

Smooth realization problem

Are there smooth versions to the objects and concepts of abstract ergodic theory?

By a smooth version we mean a C^∞-diffeomorphism of a compact manifold preserving a C^∞-measure equivalent to the volume element that is measure-isomorphic to a given automorphism.

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- Existence of volume-preserving diffeomorphisms with ergodic properties?
- What ergodic properties, if any, are imposed upon a dynamical system by the fact that it should be smooth?

Smooth realization problem

Known restrictions:

- M smooth compact manifold, $T \in \text{Diff}^{\infty}(M, \mu)$. Then: $h_{\mu}(T) < \infty$. (Kushnirenko 1965)
- In case of $M = \mathbb{S}^1$: Any diffeomorphism with invariant smooth measure is conjugated to a rotation
- In dimension $d = 2$: Weakly mixing diffeomorphisms of positive measure entropy are Bernoulli (Pesin 1977)
- No restrictions for $d > 2$ (or in case of entropy 0 for $d \geq 2$) are known!

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On the other hand: Scarcity of general results

The odometer obstacle

Smooth realization of transformations with a non-trivial odometer factor is an open problem

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PROBLEM 7.10. Find a smooth realization of:

(1) a Gaussian dynamical system with simple (Kronecker) spectrum;
(2) a dense G_δ set of minimal interval exchange transformations;
✗ (3) an adding machine;
(4) the time-one map of the horocycle flow 2.3.1 on the modular surface $SO(2)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ (which is not compact, so the standard realization cannot be used).

B. Fayad, A. Katok
Constructions in elliptic dynamics
ETDS 24 (2004), 1477-1520.

Anti-classification result for C^∞-diffeos

In a recent series of papers Foreman and Weiss extended their anti-classification result to the C^∞-setting:

Theorem (Foreman-Weiss)

Let M be either the torus $\mathbb{T}^2$, the disk $\mathbb{D}^2$ or the annulus $\mathbb{S}^1 \times [0,1]$. Then the measure isomorphism relation among pairs (S, T) of area-preserving ergodic C^∞-diffeomorphisms of M is complete analytic and hence not Borel.

von Neumann's classification problem is impossible even when restricting to smooth diffeomorphisms

Approximation by Conjugation-method: Setting

Let M be a smooth compact connected manifold of dimension $d \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ preserving a smooth volume μ,
e.g. torus $\mathbb{T}^2$, annulus $\mathbb{S}^1 \times [0,1]$ or disc $\mathbb{D}^2$ with standard circle action comprising of the diffeomorphisms $S_t(\theta, r) = (\theta + t, r)$.

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- We construct a sequence of measure-preserving diffeomorphisms

$$T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1},$$

where
$\alpha_n = \frac{p_n}{q_n} \in \mathbb{Q}$ with p_n, q_n relatively prime,
$H_n = h_1 \circ h_2 \circ ... \circ h_n$ with h_i measure-preserving diffeomorphism of M.

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  $H_n = h_1 \circ h_2 \circ ... \circ h_n$ with h_i measure-preserving diffeomorphism of M.
- We need a criterion for the aimed property expressed on the level of the maps T_n and appropriate partitions of the manifold.

Scheme

Construction of $T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1}$:

- Initial step: Choose $\alpha_0 = \frac{p_0}{q_0}$ arbitrary, $T_0 = S_{\alpha_0}$.

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  Put $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{l_n \cdot k_n \cdot q_n^2}$ with parameters $l_n, k_n \in \mathbb{Z}$.
  The conjugation map h_{n+1} and the parameter l_n are chosen such that $h_{n+1} \circ S_{\alpha_n} = S_{\alpha_n} \circ h_{n+1}$ and T_{n+1} imitates the desired property with a certain precision.

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  Then the parameter k_n is chosen large enough to guarantee closeness of T_{n+1} to T_n in the C^∞-topology:

$$\begin{aligned} T_{n+1} &= H_{n+1} \circ S_{\alpha_{n+1}} \circ H_{n+1}^{-1} \\ &= H_n \circ h_{n+1} \circ S_{\alpha_n} \circ S_{\frac{1}{l_n \cdot k_n \cdot q_n^2}} \circ h_{n+1}^{-1} \circ H_n^{-1} \\ &= H_n \circ S_{\alpha_n} \circ h_{n+1} \circ S_{\frac{1}{l_n \cdot k_n \cdot q_n^2}} \circ h_{n+1}^{-1} \circ H_n^{-1} \approx H_n \circ S_{\alpha_n} \circ H_n^{-1} = T_n \end{aligned}$$

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$\Longrightarrow$ Convergence of the sequence $(T_n)_{n \in \mathbb{N}}$ to a limit diffeomorphism with the aimed properties

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Symbolic representation of untwisted AbC-diffeomorphisms: circular systems.

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Symbolic representation of untwisted AbC-diffeomorphisms: circular systems.
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- Set $\mathcal{W}_0 = \Sigma$.
- Having built $\mathcal{W}_n$ we choose a set $P_{n+1} \subseteq (\mathcal{W}_n)^{k_n}$ of so-called *prewords* and form $\mathcal{W}_{n+1}$ by taking all words of the form

$$\mathcal{C}_n\left(w_0, w_1, \dots, w_{k_n-1}\right) = \prod_{i=0}^{q_n-1} \prod_{j=0}^{k_n-1} \left(b^{q_n-j_i} w_j^{l_n-1} e^{j_i}\right)$$

  with $w_0 \dots w_{k_n-1} \in P_{n+1}$. If $n = 0$ we take $j_0 = 0$, and for $n > 0$ we let $j_i \in \{0, \dots, q_n - 1\}$ be such that

$$j_i \equiv (p_n)^{-1}\, i \mod q_n.$$

  We note that each word in $\mathcal{W}_{n+1}$ has length $q_{n+1} = k_n l_n q_n^2$.

Circular systems

A construction sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ will be called *circular* if it is built in this manner using the $\mathcal{C}$-operators, a circular coefficient sequence and each P_{n+1} is uniquely readable in the alphabet with the words from $\mathcal{W}_n$ as letters.

Circular system

A symbolic shift $\mathbb{K}^c$ built from a circular construction sequence is called a *circular system*.

realizable as smooth diffeomorphisms using the untwisted AbC method

Functor between $\mathcal{OB}$ and $\mathcal{CB}$

Let Σ be an alphabet and $(\mathtt{W}_n)_{n\in\mathbb{N}}$ be a construction sequence for an odometer-based system with coefficients $(k_n)_{n\in\mathbb{N}}$. Then we define a circular construction sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ and bijections $c_n : \mathtt{W}_n \to \mathcal{W}_n$ by induction:

- Let $\mathcal{W}_0 = \Sigma$ and c_0 be the identity map.
- Suppose that $\mathtt{W}_n$, $\mathcal{W}_n$ and c_n have already been defined. Then we define

$$\mathcal{W}_{n+1} = \{\mathcal{C}_n\left(c_n\left(\mathtt{w}_0\right), c_n\left(\mathtt{w}_1\right), \dots, c_n\left(\mathtt{w}_{k_n-1}\right)\right) \,:\, \mathtt{w}_0\mathtt{w}_1\dots\mathtt{w}_{k_n-1} \in \mathtt{W}_{n+1}\}$$

  and the map c_{n+1} by setting

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  In particular, the prewords are

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  $$P_{n+1} = \{c_n\left(\mathtt{w}_0\right) c_n\left(\mathtt{w}_1\right) \dots c_n\left(\mathtt{w}_{k_n-1}\right) \;:\; \mathtt{w}_0\mathtt{w}_1\dots\mathtt{w}_{k_n-1} \in \mathtt{W}_{n+1}\}.$$

Functor $\mathcal{F}$

Suppose that $\mathbb{K}$ is built from a construction sequence $(\mathtt{W}_n)_{n\in\mathbb{N}}$ and $\mathbb{K}^c$ has the circular construction sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ as constructed above. Then we define a map $\mathcal{F}$ by

$$\mathcal{F}\left(\mathbb{K}\right) = \mathbb{K}^c.$$

Properties of the functor

Theorem (Foreman-Weiss 2019)

The functor $\mathcal{F}$ preserves

- weakly mixing extensions,
- compact extensions,
- factor maps,
- certain types of isomorphisms,
- the rank-one property,
- ...

M. Foreman and B. Weiss
From Odometers to Circular Systems: A Global Structure Theorem.
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Question

What other dynamical properties are preserved under $\mathcal{F}$?

Thouvenot: Does $\mathcal{F}$ preserve the loosely Bernoulli property?

Kakutani Equivalence

Let $(X, \mathcal{A}, m, T)$ and $A \in \mathcal{A}$ with $m(A) > 0$. Induced transformation T_A:

$$T_A(x) = T^{n(x)}(x), \quad \text{where} \quad n(x) = \inf \{ n \in \mathbb{N} \mid T^n(x) \in A \}.$$

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Two ergodic transformations $(X, \mathcal{A}, m, T)$ and $(Y, \mathcal{B}, \mu, S)$ are said to be *Kakutani equivalent* if there exist a set $A \in \mathcal{A}$ with $m(A) > 0$ and a set $B \in \mathcal{B}$ with $\mu(B) > 0$ such that (T_A, m_A) is isomorphic to (S_B, μ_B).

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From Abramov's entropy formula $h(T_A) = \frac{h(T)}{m(A)}$: two Kakutani equivalent automorphisms must both have entropy zero, both have finite positive entropy, or both have infinite entropy.

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From Abramov's entropy formula $h(T_A) = \frac{h(T)}{m(A)}$: two Kakutani equivalent automorphisms must both have entropy zero, both have finite positive entropy, or both have infinite entropy.
It was a long-standing open problem whether these three possibilities for entropy completely characterized Kakutani equivalence classes, but Feldman (1976): there are at least two non-Kakutani equivalent ergodic transformations in each of the three entropy classes. Ornstein-Rudolph-Weiss (1982): uncountable family

The loosely Bernoulli property

Introduced by Katok (1975) in the case of zero entropy, and, independently, by Feldman (1976) in the general case.

The loosely Bernoulli property

An ergodic automorphism T that is Kakutani equivalent to an irrational circle rotation (in the case of entropy zero) or a Bernoulli shift (in case of non-zero entropy) is said to be *loosely Bernoulli*.

Zero-entropy loosely Bernoulli automorphisms are also called *standard*.

The $\overline{f}$ distance

Suppose $T : (X, \mu) \to (X, \mu)$ is a measure-preserving automorphism and $\mathcal{P} = (P_1, \dots, P_q)$ is a finite measurable partition of X. If $b, c \in \mathbb{Z}$ with $b \leq c$, then the T-$\mathcal{P}$ *name* of a point $x \in X$ from time b to time c is the finite sequence $(a_b, a_{b+1}, \dots, a_c)$ where $T^i(x) \in P_{a_i}$ for $b \leq i \leq c$.

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$$\overline{f}_{T,\mathcal{P},n}(x, y) = 1 - (m/n),$$

where $m = \sup\{j : \text{there exist } 0 \leq k_1 < \cdots < k_j < n \text{ and } 0 \leq \ell_1 < \cdots < \ell_j < n \text{ such that } T^{k_i}x \text{ and } T^{\ell_i}y \text{ are in the same element of } \mathcal{P} \text{ for } i = 1, \ldots, j\}$.

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The $\overline{f}$ distance was used by Feldman to create a Kakutani equivalence theory for loosely Bernoulli automorphisms that parallels Ornstein's $\overline{d}$ metric and his isomorphism theory for Bernoulli shifts:

$$\overline{d}_{T,\mathcal{P},n}(x,y) = \frac{1}{n} \left| \left\{ 0 \leq i < n \mid T^i(x) \text{ and } T^i(y) \text{ are in different elements of } \mathcal{P} \right\} \right|.$$

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Example: 010101 and 101010 are $\frac{1}{6}$ apart in $\overline{f}$ while they are 1 apart in $\overline{d}$.

Loosely Bernoulli automorphisms of zero entropy

In case of zero entropy, the $\overline{f}$ distance gives a simple characterization of loosely Bernoulli automorphisms:

Characterization LB in case of zero entropy

A zero-entropy process $(T, \mathcal{P})$ is said to be loosely Bernoulli if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and a set A of measure greater than $1 - \varepsilon$ such that $\overline{f}_{T,\mathcal{P},N}(x, y) < \varepsilon$ for all $x, y \in A$.
A transformation T is LB if $(T, \mathcal{P})$ is an LB process for every finite partition $\mathcal{P}$.

Results

Theorem (Gerber-K.)

There exist

1. a loosely Bernoulli odometer-based system $\mathbb{E}$ of positive entropy such that $\mathcal{F}(\mathbb{E})$ is not loosely Bernoulli.
2. a loosely Bernoulli odometer-based system $\mathbb{K}$ of entropy zero such that $\mathcal{F}(\mathbb{K})$ is not loosely Bernoulli.
3. a non-loosely Bernoulli odometer-based system $\mathbb{L}$ of entropy zero such that $\mathcal{F}(\mathbb{L})$ is loosely Bernoulli.

Idea of proof for (2)

Alternate application of two mechanisms:

- *Feldman mechanism:* Produce arbitrary number of $(n+1)$-blocks which are almost as far apart in $\bar{f}$ as n-blocks. The construction is based on the observation that no pair of the following strings

$$ababab ab$$
$$aabbaabb$$
$$aaaabbbb$$

  can be matched very well.
- *Shifting mechanism:* Given sufficiently many $(n+1)$-blocks we can build prescribed number of $(n+p)$-blocks that are close to each other in $\bar{f}$ in the odometer-based system, while they stay $\bar{f}$ apart in the circular system

Idea of the shifting mechanism

$$\mathtt{B}_1 = \overbrace{\Big[\underbrace{[ABC]}_{n\text{-block}}[DEF][GHI]\Big]}^{(n+1)\text{-block}}\Big[[JKL][MNO][PQR]\Big]\Big[[STU][VWX][YZ\Gamma]\Big]\dots$$

$$\mathtt{B}_2 = \Big[[EFG][HIJ][KLM]\Big]\Big[[NOP][QRS][TUV]\Big]\Big[[WXY][Z\Gamma\Delta][\Theta\Lambda\Xi]\Big]\dots$$

$$\mathtt{B}_3 = \Big[[IJK][LMN][OPQ]\Big]\Big[[RST][UVW][XYZ]\Big]\Big[[\Gamma\Delta\Theta][\Xi\Pi\Sigma][\Upsilon\Phi\Psi]\Big]\dots$$

Idea of the shifting mechanism

$$
\begin{aligned}
\mathcal{B}_1 = {} & \Big[[ABC][ABC][ABC][D\boxed{\mathrm{EF}}][D\boxed{\mathrm{EF}}][D\boxed{\mathrm{EF}}][G\boxed{\mathrm{HI}}][G\boxed{\mathrm{HI}}][G\boxed{\mathrm{HI}}]\Big]\Big[[ABC][ABC][ABC][D\boxed{\mathrm{EF}}][D\boxed{\mathrm{EF}}][D\boxed{\mathrm{EF}}][G\boxed{\mathrm{HI}}][G\boxed{\mathrm{HI}}][G\boxed{\mathrm{HI}}]\Big]\dots \\
& \Big[[JKL][JKL][JKL][M\boxed{\mathrm{NO}}][M\boxed{\mathrm{NO}}][M\boxed{\mathrm{NO}}][P\boxed{\mathrm{QR}}][P\boxed{\mathrm{QR}}][P\boxed{\mathrm{QR}}]\Big]\Big[[JKL][JKL][JKL][M\boxed{\mathrm{NO}}][M\boxed{\mathrm{NO}}][M\boxed{\mathrm{NO}}][P\boxed{\mathrm{QR}}][P\boxed{\mathrm{QR}}][P\boxed{\mathrm{QR}}]\Big]\dots \\
& \Big[[STU][STU][STU][V\boxed{\mathrm{WX}}][V\boxed{\mathrm{WX}}][V\boxed{\mathrm{WX}}][Y\boxed{\mathrm{Z\Gamma}}][Y\boxed{\mathrm{Z\Gamma}}][Y\boxed{\mathrm{Z\Gamma}}]\Big]\Big[[STU][STU][STU][V\boxed{\mathrm{WX}}][V\boxed{\mathrm{WX}}][V\boxed{\mathrm{WX}}][Y\boxed{\mathrm{Z\Gamma}}][Y\boxed{\mathrm{Z\Gamma}}][Y\boxed{\mathrm{Z\Gamma}}]\Big]\dots
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_2 = {} & \Big[[\boxed{\mathrm{EF}}G][\boxed{\mathrm{EF}}G][\boxed{\mathrm{EF}}G][\boxed{\mathrm{HI}}J][\boxed{\mathrm{HI}}J][\boxed{\mathrm{HI}}J][KLM][KLM][KLM]\Big]\Big[[\boxed{\mathrm{EF}}G][\boxed{\mathrm{EF}}G][\boxed{\mathrm{EF}}G][\boxed{\mathrm{HI}}J][\boxed{\mathrm{HI}}J][\boxed{\mathrm{HI}}J][KLM][KLM][KLM]\Big]\dots \\
& \Big[[\boxed{\mathrm{NO}}P][\boxed{\mathrm{NO}}P][\boxed{\mathrm{NO}}P][\boxed{\mathrm{QR}}S][\boxed{\mathrm{QR}}S][\boxed{\mathrm{QR}}S][TUV][TUV][TUV]\Big]\Big[[\boxed{\mathrm{NO}}P][\boxed{\mathrm{NO}}P][\boxed{\mathrm{NO}}P][\boxed{\mathrm{QR}}S][\boxed{\mathrm{QR}}S][\boxed{\mathrm{QR}}S][TUV][TUV][TUV]\Big]\dots \\
& \Big[[\boxed{\mathrm{WX}}Y][\boxed{\mathrm{WX}}Y][\boxed{\mathrm{WX}}Y][\boxed{\mathrm{Z\Gamma}}\Delta][\boxed{\mathrm{Z\Gamma}}\Delta][\boxed{\mathrm{Z\Gamma}}\Delta][\Theta\Lambda\Xi][\Theta\Lambda\Xi][\Theta\Lambda\Xi]\Big]\Big[[\boxed{\mathrm{WX}}Y][\boxed{\mathrm{WX}}Y][\boxed{\mathrm{WX}}Y][\boxed{\mathrm{Z\Gamma}}\Delta][\boxed{\mathrm{Z\Gamma}}\Delta][\boxed{\mathrm{Z\Gamma}}\Delta][\Theta\Lambda\Xi][\Theta\Lambda\Xi][\Theta\Lambda\Xi]\Big]\dots
\end{aligned}
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\end{aligned}
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Real-analytic topology

Real-analytic diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity have a lift of type

$$F(x_1, x_2) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1, x_2)),$$

where the functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ are real-analytic and $\mathbb{Z}^2$-periodic for $i = 1, 2$.

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Definition

For any $\rho > 0$ we consider the set of real-analytic $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$, that can be extended to a holomorphic function on

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- $C^\omega_\rho(\mathbb{T}^2)$: set of these functions satisfying the condition $\|f\|_\rho < \infty$.
- $\mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$: set of volume-preserving diffeomorphisms homotopic to the identity, whose lift satisfies $f_i \in C^\omega_\rho(\mathbb{T}^2)$ for $i = 1, 2$.

Anti-classification result for real-analytic diffeos

Theorem (Banerjee-K)

For every $\rho > 0$ the measure-isomorphism relation among pairs (S, T) of ergodic $\text{Diff}^{\omega}_{\rho}\left(\mathbb{T}^2, \mu\right)$-diffeomorphisms is a complete analytic set and hence not Borel.

von Neumann's classification problem is impossible even when restricting to real-analytic diffeomorphisms of the torus

Thank you very much for your attention!