On some spectral problems in ergodic theory

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Spectral theory. Koopman operator

- $(X, \mathcal{B}, \mu)$ probability, standard Borel space (non-atomic!),
- $L^2(X, \mathcal{B}, \mu)$ is separable,
- $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ invertible, measure-preserving; also notation $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$,
- Koopman operator: $U_T : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$, $U_T f := f \circ T$ for $f \in L^2(X, \mathcal{B}, \mu)$,
- Spectral theory: properties of U_T, that is, properties of T that are stable under spectral isomorphism in $\mathrm{Aut}(X, \mathcal{B}, \mu)$.
- Ergodicity, weak mixing, mixing are spectral properties.[1]

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Why not Koopman on $L^p(X, \mathcal{B}, \mu)$? :-(

- $p \neq 2$
- $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, $S \in \mathrm{Aut}(Y, \mathcal{C}, \nu)$ are ergodic, $V : L^p(X, \mathcal{B}, \mu) \to L^p(Y, \mathcal{C}, \nu)$ an isometry such that $V \circ U_T = U_S \circ V$.

Proposition

Under the above assumptions T and S are measure-theoretically isomorphic.

- Use Lamperti's theorem to obtain that $(Vf)(y) = j(y) \cdot f(Jy)$, where $J : Y \to X$ is non-singular.
- Equivariance yields: $j(y)f(TJy) = j(Sy)f(JSy)$; take $f = 1$, to obtain that $j = const$ by the ergodicity of S.
- Notice that the image $J_*(\nu)$ is a T-invariant, ergodic measure satisfying $J_*(\nu) \ll \mu$, and use ergodicity of T to conclude.

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- (Thouvenot's problem, 1986) Is is true that for each ergodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ there exists $f \in L^1(X, \mathcal{B}, \mu)$ such that $L^1(X, \mathcal{B}, \mu) = \overline{\mathrm{span}}\{f \circ T^k : k \in \mathbb{Z}\}$ for some $f \in L^1(X, \mathcal{B}, \mu)$?
- (Iwanik, 1991) For each Bernoulli automorphism T and $p > 1$, each $n \geqslant 1$ and all $f_1, \ldots, f_n \in L^p(X, \mathcal{B}, \mu)$, we have

$$\overline{\mathrm{span}}\{f_j \circ T^k : k \in \mathbb{Z}, j = 1, \ldots, n\} \neq L^p(X, \mathcal{B}, \mu)$$

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Classification of unitary operators on separable Hilbert spaces. I

Theorem (Herglotz)

If $U : H \to H$ is a unitary operator on a Hilbert space H and $f \in H$, then the sequence $\mathbb{Z} \ni k \mapsto \langle U^k f, f \rangle$ is positive definite and theorefore there exists a unique (positive, Borel) measure σ_f on $\mathbb{S}^1$ such that

$$\widehat{\sigma}_f(k) := \int_{\mathbb{S}^1} z^k \, d\sigma_f(z) = \langle U^k f, f \rangle \text{ for each } k \in \mathbb{Z}.$$

- σ_f is called the *spectral measure* of f.

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Classification of unitary operators on separable Hilbert spaces. II

- $H_\sigma := L^2(\mathbb{S}^1, \sigma)$, where σ is a (positive, Borel) finite measure on the circle,
- $V_\sigma : H_\sigma \to H_\sigma$; $(V_\sigma f)(z) = zf(z)$ is unitary,
- $H_\sigma = \mathbb{Z}(\mathbb{1}) := \overline{\mathrm{span}}\{V_\sigma^k \mathbb{1} : \ k \in \mathbb{Z}\}$ – one says that H_σ is equal to the cyclic space $\mathbb{Z}(\mathbb{1})$, where $\mathbb{1}(z) = 1$.
- V_σ is an example of a unitary operator (defined on a separable Hilbert space) with *simple spectrum*.
- If H is a (separable) Hilbert space and U is a unitary operator on it such that $H = \mathbb{Z}(f)$ for some $f \in H$, then the map

$$U^k f \mapsto V_{\sigma_f}^k(\mathbb{1}) = z^k$$

extends to a (linear) isometry intertwining U and V_{σ_f}.
- If above holds, we say that U has *simple spectrum*.

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- $U : H \to H$ unitary on a separable Hilbert space H.

Spectral theorem

There exists a decomposition, called a spectral decomposition, $H = \bigoplus_{n\geqslant 1} \mathbb{Z}(f_n)$ such that
(*) $\sigma_{f_1} \gg \sigma_{f_2} \gg \dots$ (spectral sequence).
For any other spectral decomposition $H = \bigoplus_{n\geqslant 1} \mathbb{Z}(f_n')$, we have
(**) $\sigma_{f_n} \equiv \sigma_{f_n'}$ for each $n \geqslant 1$.

- (the type of) σ_{f_1} is called the *maximal spectral type* of U and is denoted by σ_U.
- $M_U(z) := \sum_{n\geqslant 1} \mathbb{1}_{\mathrm{supp}\frac{d\sigma_{f_n}}{d\sigma_{f_1}}}(z)$ is called the *spectral multiplicity* function of U (it is defined σ_U-a.e.).
- Any sequence $\sigma_1 \gg \sigma_2 \dots$ can be realized as a spectral sequence of some U.
- Two unitary operators are (spectrally) isomorphic if and only if they have the same spectral sequence (*) (up to equivalence od spectral measures).

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- Which sequences $\sigma_1 \gg \sigma_2 \gg \ldots$ appear as spectral sequences of Koopman operators $U_T|_{L_0^2(X,\mathcal{B},\mu)}$ for ergodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$?

(A) What measures appear as maximal spectral type of an ergodic automorphism?

(B) What subsets of $\mathbb{N} \cup \{\infty\}$ appear as the set $\mathrm{essval}(M_T)$ of essential values of an ergodic automorphism T? (Such sets are called *Koopman realizable*).

Examples: (i) T Bernoulli: $\lambda \equiv \lambda \equiv \ldots$, so $\sigma_{U_T} = \sigma_T = [\lambda]$, $\mathrm{essval}(M_T) = \{\infty\}$ (all Bernoulli automorphisms are spectrally isomorphic).
(ii) T irrational rotation: $\sigma \gg 0 \equiv 0 \equiv \ldots$, where $\sigma_T = \sigma = \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} \delta_{e^{2\pi i \ell \alpha}}$ and $\mathrm{essval}(M_T) = \{1\}$ (ergodic automorphisms with discrete spectrum have simple spectrum).

- Which sequences $\sigma_1 \gg \sigma_2 \gg \dots$ appear as spectral sequences of Koopman operators $U_T|_{L^2_0(X,\mathcal{B},\mu)}$ for ergodic $T \in \mathrm{Aut}(X,\mathcal{B},\mu)$?

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Maximal spectral type. I

Proposition (Known restrictions:)

(i) Topological support $\mathrm{supp}(\sigma_T)$ of σ_T is $\mathbb{S}^1$.
(ii) The measure $\widetilde{\sigma_T}(A) := \sigma_T(\overline{A})$ is equivalent to σ_T.
(iii) If $e^{2\pi i\alpha}$ is an eigenvalue of U_T then the measure $\sigma_{T,i}(A) := \sigma_T(e^{2\pi i\alpha} \cdot A)$ is equivalent to σ_T.

- Use $\mathrm{supp}(\sigma_T) = \{z \in \mathbb{C} : z \cdot Id - U_T$ is not a bijection$\} = \{z \in \mathbb{C} :\ z$ is an approximative eigenvalue$\}$; to solve $\|U_T(f) - zf\|_{L^2} < \epsilon$ use Rokhlin lemma $f = \sum_{i=0}^{h-1} z^i \mathbb{1}_{T^iF}$.

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Maximal spectral type. II

- If σ is a continuous symmetric measure on the circle and $U = e^{V_\sigma} := \bigoplus_{\ell\geqslant 0} V_\sigma^{\odot\ell}$, then U is a Koopman operator via the classical Gaussian construction. Indeed, take a centered, real stationary Gaussian process $(X_n)_{n\in\mathbb{Z}}$ with spectral measure σ, i.e. $\mathbb{E}(X_n\cdot X_0)=\widehat{\sigma}(n)$ for each $n\in\mathbb{Z}$, with joint distribution μ_σ on $\mathbb{R}^{\mathbb{Z}}$ and consider the shift T on $(\mathbb{R}^{\mathbb{Z}},\mu_\sigma)$.
- $\sigma_U=\sum_{n\geqslant 1}\frac{1}{2^n}\sigma^{*n}$ is a measure of maximal spectral type for an ergodic (in fact, weakly mixing) automorphism.
- (Girsanov's theorem, 1950th) Either $M_U=1$ or M_U has to be unbounded. We can have $\mathrm{essval}(M_U)=\{1,\infty\}$ (take $\sigma\perp\lambda$ with $\sigma*\sigma\equiv\lambda$), otherwise this set is infinite and probably has interesting arithmetic properties. Danilenko and Ryzhikov in 2010 proved that every multiplicative sub-semigroup of $\mathbb{N}$ has a a Gaussian "realization".
- Not every continuous, symmetric measure can be realized as a maximal spectral type of a Koopman operator. Indeed, we can find a so called Kronecker measure σ (continuous, with full topological support) satisfying: For each $f\in L^2(\mathbb{S}^1,\sigma)$ and $\epsilon>0$ there exists $k\in\mathbb{Z}$ such that $\|f-z^k\|_{L^2(\sigma)}<\epsilon$. Then the famous Foiaș-Stratila theorem from 1967 tells us that whenever $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ is ergodic and a real-valued $f\in L^2(X,\mathcal{B},\mu)$ has $\sigma_f\equiv\sigma$ then the process $(f\circ T^n)_{N\in\mathbb{Z}}$ has to be Gaussian; then f will be in the first chaos of the corresponding L^2-space and also $\sigma\perp\sum_{j\geqslant 2}\frac{1}{2^j}\sigma^{*j}$, so σ cannot be a measure of maximal spectral type.

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Maximal spectral type. III

OUR KNOWLEDGE ABOUT THE MAXIMAL SPECTRAL TYPE HAS NOT CHANGED SINCE THE 1960TH!!

Maximal spectral multiplicity

- Oseledets (1966): There exists an ergodic T such that $1 < \mathrm{esssup}(M_T) < \infty$. [2]
- Robinson (1983): For each $n \geqslant 1$, there exists an ergodic T such that $\mathrm{esssup}(M_T) = n$. [3]

[2] Double group extensions of IETs.

[3] Double group extensions, Katok-Stepin theory of periodic approximation to apply a generic type arguments.

Essential values of the multiplicity function

- $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$; $\mathrm{essval}(M_T)$ stands for the essential values of the multiplicity function M_T.
- General multiplicity conjecture: Each subset $E \subset \mathbb{N}$ is realizable as $\mathrm{essval}(M_T)$.

Recall: Generally, we are interested which sequences $\sigma_1 \gg \sigma_2 \gg \ldots$ are realizable as spectral sequences of Koopman operators. In this sequence we have either $\equiv$ or $\gg$ (without equivalence).

Reformulation of spectral multiplicity conjecture

Is it true that each sequence (finite or infinite) $(s_n)_{n\geq 1} \in \{\equiv, \gg\}^{\mathbb{N}}$ is Koopman realizable?

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Is it true that each sequence (finite or infinite) $(s_n)_{n \geqslant 1} \in \{\equiv, \gg\}^{\mathbb{N}}$ is Koopman realizable?

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Theorem (Robinson (1986), Goodson-Kwiatkowski-L.-Liardet (1992), Kwiatkowski (jr.)-L. (1995) with final result)

Each subset $1 \in E \subset \mathbb{N}$ is Koopman realizable, i.e. there exists an ergodic T such that $\mathrm{essval}(M_T) = E$.

- How to create symmetry? $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, G a compact, Abelian group, $\phi : X \to G$ a cocycle, one considers

$$T_\phi : X \times G \to X \times G, \;\; T_\phi(x, g) = (Tx, \phi(x) + g).$$

- $L^2(X \times G, \mu \otimes m_G) = \sum_{\chi \in \widehat{G}} L^2(X, \mu) \otimes \chi$. Note that $U_{T_\phi}(L^2(X, \mu) \otimes \chi) = L^2(X, \mu) \otimes \chi$.
- Assume that $S \in C(T)$ and for some $v \in \mathrm{Aut}(G)$ we can solve the equation $\phi(Sx) - v(\phi(x)) = \xi(Tx) - \xi(x)$ for a measurable $\xi : X \to G$.
- $S_{\xi,v}(x, g) := (Sx, \xi(x) + v(g))$ is an element of the centralizer of T_ϕ.
- $U_{S_{\xi,v}}(L^2(X, \mu) \otimes \chi) = L^2(X, \mu) \otimes (\chi \circ v)$.
- The lengths of the orbits of v on $\widehat{G}$ yields a lower bound on the multiplicity.
- Passing to $T_{\phi H}$ for a closed subgroup $H \subset G$ yields LESS subspaces of the form $L^2(X, \mu) \otimes \chi$ (algebra!) under consideration.

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Rokhlin's homogeneous spectrum problem

Is it true that for each $n \geqslant 2$ there is an ergodic automorphism T such that $\mathrm{essval}(M_T) = \{n\}$?

- Katok (mid 1980): For a generic automorphism T, $\mathrm{essval}(M_{T\times T}) \subset \{2, 4\}$ (via Katok's linked approximation theory).
- Katok's conjecture: Generically, we have $\mathrm{essval}(M_{T^{\times n}}) = \{n, n(n-1), \ldots, n!\}$ (note that for $n = 2$ it yields positive answer to Rokhlin's question).
- Proof of Katok's conjecture: Ageev (1999) - general case, Ryzhikov (1999) - $n = 2$.
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Theorem (Katok-L., 2009)

Each finite set $2 \in E \subset \mathbb{N}$ is Koopman realizable.

- Obtained through studying isometric extensions of Cartesian squares.
- Exploiting the technique of weak limits to obtain simplicity of the spectrum of tensor products operators of the form $e^V \otimes W$ with a simultaneous control of homogenous multiplicity for the $W \otimes W$.

Remark (i) The above theorem was extended by Danilenko to all $2 \in E \subset \mathbb{N}$ in 2010.
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- symmetry of double skew products with a group structure in the second extension, first noticed by Oseledets, originally systematically explored by Robinson and further developed Goodson-Kwiatkowski-L.-Liardet,
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Comments and questions from Katok-L. (2009). Maximal spectral type

Problem 1: Can the maximal spectral type be absolutely continuous but not Lebesgue?

Problem 2: Can the maximal spectral type σ for U_T be absolutely continuous with respect to its convolution $\sigma * \sigma$ but not equivalent to it? [5]

Notice that there are three known possibilities:

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- Frączek proved it for some group extensions of rotations.

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- Horocycle flows have Lebesgue spectrum of infinite multiplicity (Parasyuk, 1953).
- Smooth time changes of them are mixing (Kushnirenko, 1974, Marcus, 1977).

Conjecture (Katok, Thouvenot; 2006): All flows obtained by a sufficiently smooth time change of horocycle flows have countable Lebesgue spectrum.

- Maximal spectral type Lebesgue was proved by Forni and Ulcigrai in 2012, and (the absolute continuity of the maximal spectral type) independently by Tiedra de Aldecoa (2012).
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- Still open...
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