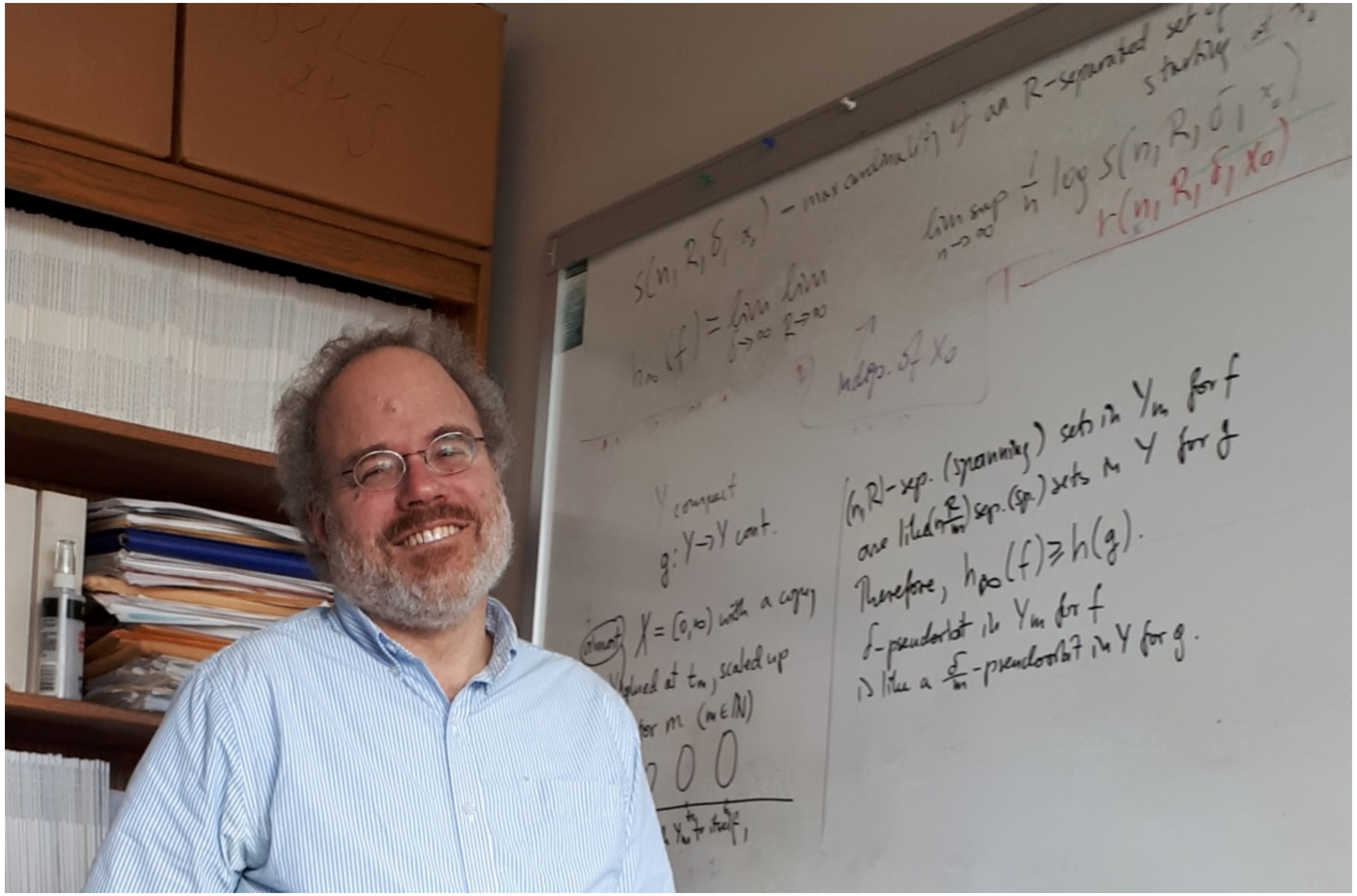
COARSE ENTROPY

MICHAŁ MISIUREWICZ

Indiana University-Purdue University Indianapolis

This is joint work with William Geller



We consider a version of coarse geometry that came from the geometric group theory. Thus, we work with metric spaces, but we are interested only in what happens in *large scale*. We look at our space from further and further away. In particular, any given bounded set looks like a singleton.

Let X and Y be metric spaces. We will use d for the metric in both spaces. A map $\varphi : X \to Y$ is called a *quasi-isometric embedding* if there exist positive constants L and C, such that

$$\frac{1}{L}d(x,y) - C \leq d(\varphi(x), \varphi(y)) \leq Ld(x,y) + C$$

for all points $x, y \in X$. Note that φ does not have to be continuous. If additionally there is a positive constant M such that $\varphi(X)$ is M-dense in Y, that is, for every $y \in Y$ there is $x \in X$ such that $d(\varphi(x), y) \leq M$, then φ is called a *quasi-isometry*. We say then that Y is *quasi-isometric* to X.

If $\xi, \psi : X \to Y$ are maps, we will say that they are *almost equal* if there exists a constant M such that $d(\xi(x), \psi(x)) \leq M$ for every $x \in X$.

A quasi-isometry $\varphi : X \to Y$ has a *quasi-inverse*, that is, a map $\psi : Y \to X$ such that $\psi \circ \varphi$ is almost equal to the identity on X and $\varphi \circ \psi$ is almost equal to the identity on Y. This map ψ is also a quasi-isometry.

The relation of being quasi-isometric is an equivalence relation. Quasi-isometries play the same role in coarse geometry as isometries in geometry or homeomorphisms in topology. A simple example of a quasi-isometry is the embedding $\varphi : \mathbb{Z} \to \mathbb{R}$. Thus, $\mathbb{Z}$ and $\mathbb{R}$ are quasi-isometric.

If φ satisfies only the second inequality in the definition of a quasi-isometric embedding, that is,

$$d(\varphi(x), \varphi(y)) \leq Ld(x, y) + C,$$

then we will say that f is *coarse Lipschitz.*

Dynamical system: we iterate a map $f : X \to X$. It is not clear what we should assume about f, so we do not assume anything.

If $g : Y \to Y$ is a map, we will say that f is *quasi-embedded* in g if there exists a quasi-isometric embedding $\varphi : X \to Y$ such that $\varphi \circ f$ is almost equal to $g \circ \varphi$.

How to define quasi-conjugacy? A natural idea would be to say that f is quasi-conjugate to g if f is quasi-embedded in g via a quasi-isometry. Unfortunately, this does not work.

Example 1. Let $X = \mathbb{Z}$ and $Y = \mathbb{R}$ and let $f : X \to X$ and $g : Y \to Y$ be defined by the same formula $x \mapsto x^2$. If $\varphi : X \to Y$ is the natural embedding, $\varphi(x) = x$, then clearly φ is a quasi-isometry and $\varphi \circ f = g \circ \varphi$. However, there is no quasi-isometry $\psi : Y \to X$ for which $\psi \circ g$ is almost equal to $f \circ \psi$.

Definition 1. The maps $f : X \to X$ and $g : Y \to Y$ are *coarsely conjugate* if there exists a quasi-isometry $\varphi : X \to Y$ and its quasi-inverse $\psi : Y \to X$ such that $\varphi \circ f$ is almost equal to $g \circ \varphi$ and $\psi \circ g$ is almost equal to $f \circ \psi$.

Coarse conjugacy defined in this way is an equivalence relation.

Note that if f and g are conjugate (in the classical sense) they are not necessarily quasi-conjugate.

Proposition 1. *Consider maps* $f : X \to X$ *and* $g : Y \to Y$ *for which there exists a quasi-isometry* $\varphi : X \to Y$ *such that* $\varphi \circ f$ *is almost equal to* $g \circ \varphi$ *and* g *is coarse Lipschitz. Then* f *is also coarse Lipschitz and for any quasi-inverse* ψ *of* φ *the maps* f *and* g *are coarsely conjugate via* φ *and* ψ*.*

Example 2. Take $X = Y = [2, \infty)$, $f(x) = x^2$, $g(x) = x^2 + \frac{1}{x}$, and both φ and ψ equal to the identity. clearly, the pair (φ, ψ) is a coarse conjugacy between f and g. However, it is not a coarse conjugacy between f^2 and g^2. Nevertheless, f^2 and g^2 are coarsely conjugate via $\varphi'(x) = x - \frac{1}{2x^2}$ and $\psi'(x) = x + \frac{1}{2x^2}$.

Conjecture 1. If f and g are coarsely conjugate then so are f^n and g^n for all natural n.

Lemma 1. *If f and g are coarsely conjugate and g is coarse Lipschitz, then for any natural n the maps f^n and g^n are coarsely conjugate via the same quasi-isometries as f and g.*

Bowen's definition of topological entropy in the usual case, when $f : X \to X$ is a continuous map:

(Piece of) trajectory of length n: a finite sequence $(x, f(x), f^2(x), \dots, f^n(x))$. Distance between $(x_0, x_1, x_2, \dots, x_n)$ and $(y_0, y_1, y_2, \dots, y_n)$ is $\max_i d(x_i, y_i)$.

$s(f, n, \varepsilon)$ is the supremum of the cardinalities of ε-separated sets of orbits of f of length n; $r(f, n, \varepsilon)$ is the infimum of the cardinalities of ε-spanning sets of orbits of f of length n.

Topological entropy of f:

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(f, n, \varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, \varepsilon).$$

We look through a better and better binocular and note the exponential growth rate of distinguishable orbits of length n as $n \to \infty$.

Instead of orbits we may use δ-pseudoorbits. A δ-pseudoorbit of f of length n starting at x_0 is a sequence $(x_0, x_1, \ldots, x_n)$ such that $d(f(x_i), x_{i+1}) \leq \delta$ for $i = 0, 1, \ldots, n-1$. Let $s(f, n, \varepsilon, \delta)$ be the supremum of the cardinalities of ε-separated sets of δ-pseudoorbits of f of length n.

Theorem 1 (M., 1986)**.** *If X is a compact metric space and $f : X \to X$ a continuous map, then*

$$h(f) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(f, n, \varepsilon, \delta).$$

In the coarse case, using δ-pseudoorbits is natural, since if we look from far away, we do not see exactly the location of the points. Instead of looking from far away we may look through a better and better binocular, but from the other side of it. This means that ε (which we now rename R) goes to infinity, rather than to zero.

Moreover, since our space is not bounded (otherwise the situation is trivial), we have to fix the starting point x_0 of the δ-pseudoorbits.

Definition of the coarse entropy of f:

$$h_\infty(f) = \lim_{\delta\to\infty} \lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log s(f, n, R, \delta, x_0),$$

The value of $h_\infty(f)$ does not depend on the choice of x_0.

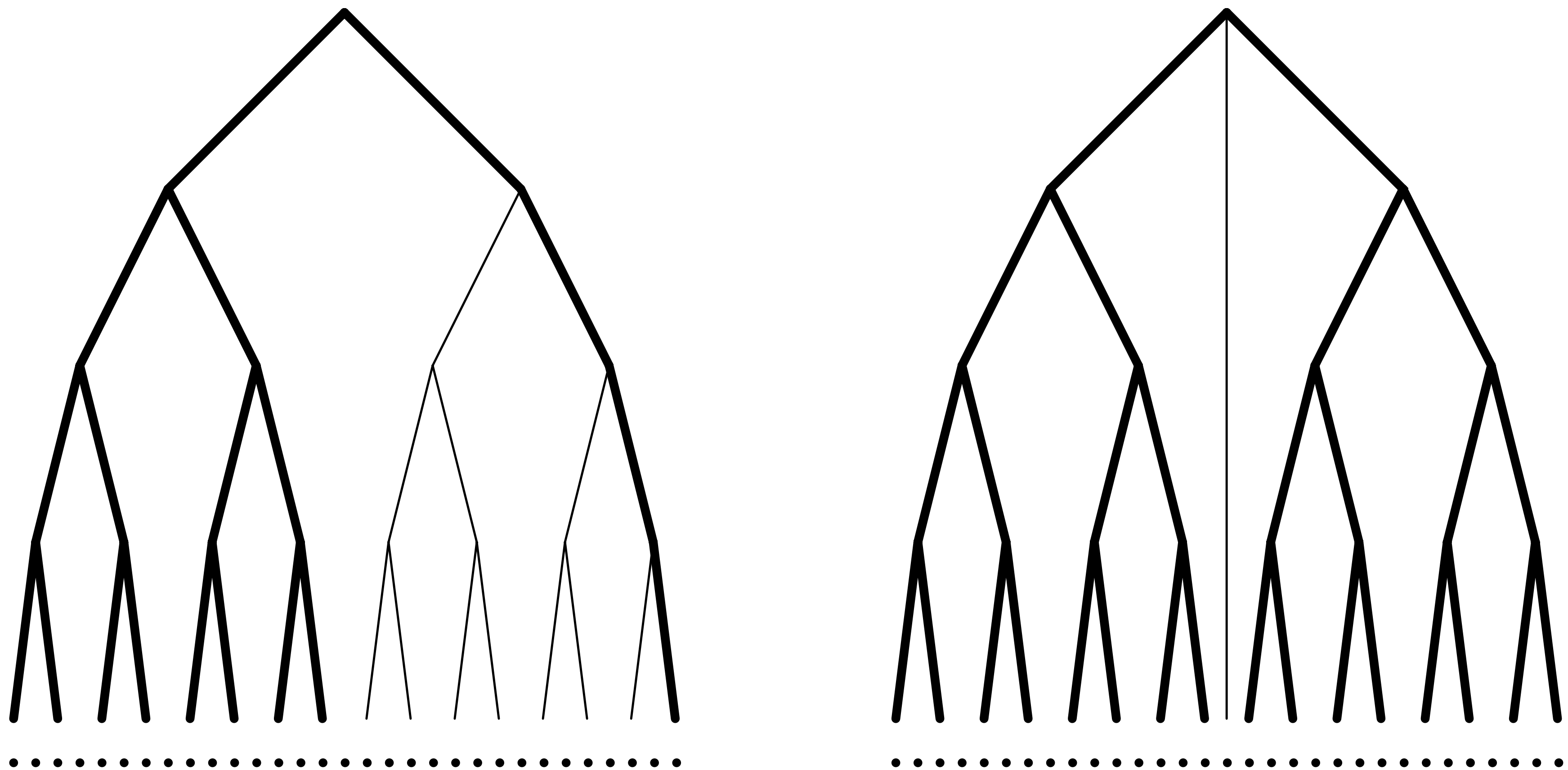
Theorem 2. *We have*

$$h_\infty(f) = \lim_{\delta\to\infty} \lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log r(f, n, R, \delta, x_0),$$

Theorem 3. *If f is quasi-embedded in g then $h_\infty(f) \leq h_\infty(g)$.*

Corollary 1. *If f is quasi-embedded in g and g is quasi-embedded in f then $h_\infty(f) = h_\infty(g)$. Therefore, the coarse entropy is an invariant of quasi-conjugacy. In particular, if we change the metric d in the phase space to a metric that is bi-Lipschitz equivalent to d, the coarse entropy will not change.*

Note that if there are quasi-isometric embeddings from X to Y and from Y to X, it does not mean that X and Y are quasi-isometric. Therefore, if f is quasi-embedded in g and g is quasi-embedded in f then it may happen that f and g are not quasi-conjugate (take the identities in X and Y).



Example 3. This is an example where f and g are homeomorphisms, they are conjugate via a Lipschitz continuous homeomorphism φ (that is, $\varphi \circ f = g \circ \varphi$), but $h_\infty(g) > h_\infty(f)$.

Let $X = Y$ be the half-plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Let $f : X \to X$ be given by the formula $f(x, y) = (2x, y)$. We have $h_\infty(f) \leq \log 2$.

The map $\varphi : X \to Y$ maps each horizontal line $H_t = \{(x, y) \in \mathbb{R}^2 : y = t\}$ to itself by squeezing linearly the segment (in the variable x) $[-e^t, e^t]$ to the segment $[-1, 1]$ and translating the remaining two half-lines. Thus, if $-e^y \leq x \leq e^y$, then $\varphi(x, y) = (xe^{-y}, y)$; if $x > e^y$ then $\varphi(x, y) = (x - e^y + 1, y)$; and if $x < -e^y$ then $\varphi(x, y) = (x + e^y - 1, y)$. Clearly, φ is a homeomorphism.

We set $g = \varphi \circ f \circ \varphi^{-1}$. Then $h_\infty(g) = \infty$.

Theorem 4. *For any $k \geq 1$ we have $h_\infty(f^k) \leq k h_\infty(f)$. If additionally f is coarse Lipschitz, then $h_\infty(f^k) = k h_\infty(f)$.*

Example 4. This is an example that in the above theorem, if we do not make any additional assumptions, then it can happen that $h_\infty(f^k) < kh_\infty(f)$.

Let X be a disjoint union of rectangles P_n, $n = 0, 1, 2, \dots$. Rectangle P_{2m} has size 1×2^m and rectangle P_{2m+1} has size $2^m \times 1$. Let c_n be the center of the rectangle P_n. On each rectangle the metric is the maximum of horizontal and vertical distances. If $x \in P_n$ and $y \in P_m$ for $n < m$, then

$$d(x, y) = d(x, c_n) + d(y, c_m) + (n + 1) + (n + 2) + \cdots + m$$

(that is, the distance between P_n and P_{n+1} is $n + 1$).

The map f maps P_n onto P_{n+1} by a linear map that preserves the horizontal and vertical directions. Thus, as we apply f repeatedly, the rectangles get alternately stretched horizontally while contracted vertically, and stretched vertically while contracted horizontally. However, f^2 only stretches each rectangle in one direction by factor 2.

We have $h_\infty(f^2) \le \log 2 < 2 \log 2 \le 2h_\infty(f)$.

Theorem 5. *Let $f : X \to X$ and $g : Y \to Y$ be maps. Then*

$$h_\infty(f \times g) \leq h_\infty(f) + h_\infty(g).$$

Example 5. This example shows that even if we assume that if f and g increase distances at most 2 times and do not decrease distances, we may not get equality in Theorem 5.

We define the spaces X and Y in a similar way as in Example 4, except that instead of rectangles, we take segments of the real line. The point c_n will be the left endpoint of the nth segment, and the distance in the space is defined in a similar way as in Example 4. The length of the zeroth segment is 1. The lengths of the next segments will be determined by the maps f and g. Both of them map the nth segment onto the $(n+1)$st one in a linear way; it will be multiplication by 1 or 2. If $2^{k^2} \leq n < 2^{(k+1)^2}$, then if k is even then f multiplies by 1 and g by 2; if k is odd then f multiplies by 2 and g by 1.

We have $h_\infty(f) \geq \log 2$ and $h_\infty(g) \geq \log 2$, while $h_\infty(f \times g) \leq \log 2$.

Lemma 2. *If $f : \mathbb{R}^q \to \mathbb{R}^q$ is a linear map with all eigenvalues of absolute value larger than 1 and the absolute value of the determinant of f is Λ, then $h_\infty(f) = \log \Lambda$.*

Lemma 3. *If $f : \mathbb{R}^q \to \mathbb{R}^q$ is a Lipschitz continuous map with Lipschitz constant $\lambda > 1$ then $h_\infty(f) \leq q \log \lambda$.*

Remark 1. If instead of assuming in Lemma 3 that f is Lipschitz, we assume only that it is coarse Lipschitz, the result will be the same.

Theorem 6. *If $f : \mathbb{R}^q \to \mathbb{R}^q$ is a linear map, then $h_\infty(f) = \log \Lambda$, where Λ is the absolute value of the product of all eigenvalues of f that have absolute value larger than 1.*

Example 6. Let X be the space l_∞ of bounded real sequences, with the sup norm, and let $f : X \to X$ be the identity map. Fix $\delta, R > 0$. Let x_0 be the zero sequence. If $n \geq R/\delta$ then for every k there exists a δ-pseudoorbit of length n starting at x_0 and ending at the sequence whose only non-zero term is the kth one, and it is equal to R. The set of those δ-pseudoorbits is an R-separated set of cardinality infinity. This proves that $h_\infty(f) = \infty$.

The above example, and easy to construct similar ones, is based on the property of the space X that for every R there are bounded sets with R-separated infinite subsets. However, there is an example of a space where the closure of every bounded set is compact, so every R-separated subset of a bounded set is finite, but nevertheless the identity has coarse entropy infinity.

Box counting dimension $\mathrm{BCD}(A)$ of a set A is the limit (if it exists)

$$\mathrm{BCD}(X) = \lim_{\varepsilon \to 0} \frac{\log r(X, \varepsilon)}{-\log \varepsilon},$$

where $r(X, \varepsilon)$ is the minimum of cardinalities of ε-spanning subsets of X.

Example 7. Let $\mathbb{S}^{q-1}$ be the unit sphere in $\mathbb{R}^q$. Let $A \subset \mathbb{S}^{q-1}$ be a set having box-counting dimension. Set

$$X = \{tx \in \mathbb{R}^q : t \geq 0, \ x \in A\}.$$

Take $\lambda > 1$ and define $f : X \to X$ by $f(x) = \lambda x$. Then

$$h_\infty(f) = (\mathrm{BCD}(A) + 1) \log \lambda.$$