C^1 actions of the mapping class group on S^1

Kamlesh Parwani

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dedicated to the memory of A. Katok

Upside down Klein

Turn the Erlangen Program on its head and instead study groups by the manifolds they act on.
Zimmer proposed studying the actions of large groups on small manifolds. Mapping class groups are "large" and one-manifolds are "small".
Problem: Fix a manifold. Show that there exists a n such that if the genus of the surface is at least n, then the mapping class group of the surface does not act effectively on the manifold.

Old results

Theorem (Hölder)

Let G be a finitely generated subgroup of the orientation preserving homeomorphisms of the real line. If G acts freely and effectively on a closed subset of $\mathbb{R}$, then G is abelian.

Corollary

Let G be a finitely generated subgroup of the orientation preserving homeomorphisms of the circle. If G acts freely and effectively on a closed subset of the circle, then G is abelian.

C^2 is not the same as C^1

Theorem (Plante and Thurston)

If G is a finitely generated nilpotent subgroup of C^2 orientation preserving diffeomorphisms of the circle, G is abelian.

Theorem (Farb and Franks)

The group of C^1 orientation preserving diffeomorphisms of the circle contains all finitely generated nilpotent groups.

C^1 is not like C^0

Theorem (Thurston)

Let S be a surface with negative Euler characteristic and exactly one puncture or boundary component. The mapping class group of S is a subgroup of the orientation preserving homeomorphisms of the circle.

Theorem (Parwani)

Let S be a surface of genus at least 4, finitely many boundary components and punctures. The mapping class group of S is a not a subgroup of the C^1 orientation preserving diffeomorphisms of the circle.

C^0 is the same as Lipschitz

Theorem (Deroin, Klepstyn, Navas)

Let G be a subgroup of the orientation preserving homoemorphisms of a one-manifold. Then the action of G is conjugate to a Lipschitz action.

Thurston's Action

Thurston has shown that mapping class groups of surfaces with boundary have effective/faithful $\underline{C^0 \text{ actions}}$ on the circle.

Start by picking representatives of $Mod(S)$ on S and then lift these to the Poincaré disc by choosing the lifts that fix a particular lift of the puncture on the circle at infinity. Since the action on the circle at infinity is the same up to isotopy, this defines an action of $Mod(S)$ on the circle at infinity.

The goal here is to show that this is not possible for $\underline{C^1 \text{ actions}}$.

Mapping class group $Mod(S)$

Let S be a surface of genus at least 4 (with or without boundary) and let $Mod(S)$ be the mapping class group of S.

Theorem [$Mod(S)$]

There are no effective C^1 actions of $Mod(S)$ on the circle. In fact, if genus is at least 6, every C^1 action of $Mod(S)$ on the circle is trivial.

General result

Theorem [General result]

Let H and G be finitely generated groups that have trivial first cohomology. Then for any C^1 action of $H \times G$ on the circle, either the action of subgroup $H \times 1$ or the subgroup $1 \times G$ factors through an action of a finite abelian group.

Corollaries

Let F_n be the free group with n generators.

Corollary [$Aut(F_n)$]

There are no effective C^1 actions of $Aut(F_n)$ or $Out(F_n)$, where $n \geqslant 6$ on the circle.

Corollary [Kazhdan groups]

Let H and G be Kazhdan groups with property T. There are no effective C^1 actions of $H \times G$ on the circle.

Kazhdan groups with property T have trivial first cohomology.

Theorem (Navas)

Let G be a Kazhdan group acting on the circle. If the action of G is $C^{1+1/2}$, it is trivial.

Related Foliations

There exists a closed 4-manifold with fundamental group equal to the mapping class group of a surface of genus 6. So we obtain a codimension one foliation on a closed 5-manifold via the suspension construction.

This foliation is Lipschitz and not homeomorphic to a C^1 foliation.

Other examples of non-smoothable C^1 foliations have been provided by Pixton and Cantwell & Conlon.

Trivial cohomology (just like lattices)

Theorem (Dehn and others)

If S is a surface of genus greater than or equal to 2, then $H^1(Mod(S), \mathbb{R})$ is trivial.

There are no nontrivial homomorphisms from $Mod(S)$ into $\mathbb{R}$.

Thurston Stability Lemma

Theorem (Thurston)

Let G be a finitely generated group acting on $\mathbb{R}^n$ with a global fixed point x. If the action is C^1 and $Dg(x)$ is the identity for all $g \in G$, then either there is a nontrivial homomorphism of G into $\mathbb{R}$ or G acts trivially.

Idea of the proof

Use commuting elements to find a fixed point (the hard part) and then apply Thurston's Stability Lemma.
So you have to first check that the action is C^1-flat at a fixed point.

Thurston Stability Lemma in dimension one

Theorem

Let G be a finitely generated group acting on $\mathbb{R}$ with a global fixed point x. If the action is C^1 ~~and $Dg(x)$ is the identity for all $g \in G$~~, then either there is a nontrivial homomorphism of G into $\mathbb{R}$ or G acts trivially.

Müller-Tsuboi trick

When you conjugate by $e^{\frac{-1}{x}}$, the C^1 diffeomorphism can by conjugated to a C^1-flat diffeomorphism.
So you really don't have to check that the action is C^1-flat.

Theorem (Parwani)

Let G be a finitely generated group acting on S^1 with a periodic orbit. If the action is C^1, then either there is a nontrivial homomorphism of G into $\mathbb{R}$ or G acts trivially.

Hyperbolic fixed points

Theorem (Deroin, Kleptsyn, and Navas)

Let G be a finitely generated group with a C^1 action on the circle. Then there is a G-invariant probability measure or there is an element $g \in G$ such that the nonempty fixed point set of g consists entirely of hyperbolic fixed points only.

Mean rotation homomorphism

Let μ be a G-invariant probability measure.

The map $\rho : G \to \mathbb{R}$ defined by

$$g \to \int_{S^1} (\tilde{g} - Id)\, d\tilde{\mu} \pmod 1,$$

where $\tilde{g}$ and $\tilde{\mu}$ are lifts of g and μ to the real line, is a homomorphism.

If $\rho(g) = 0$, then g has a fixed point.

Sketch of the proof

Let S be a surface of genus at least 6. So S can be split into two subsurfaces S_1 and S_2 such that each subsurface is of genus at least 3.

This insures that the mapping class groups $Mod(S_1)$ and $Mod(S_2)$ have trivial homology.

Also, note that $Mod(S_1)$ and $Mod(S_2)$ commute and are subgroups of $Mod(S)$. We will show that either $Mod(S_1)$ acts trivially or $Mod(S_2)$ acts trivially.

There is a $Mod(S_1)$-invariant measure

Consider the mean rotation homomorphism into $\mathbb{R}$. This must be trivial. So every element of $Mod(S_1)$ acts with a fixed point. Then there must be a global fixed point.

Apply the Thurston Stability Lemma. Since every homomorphism into $\mathbb{R}$ must be trivial, the action of $Mod(S_1)$ on the circle is trivial.

Hyperbolic fixed points

There is an element $g \in Mod(S_1)$ that acts with hyperbolic fixed points. In particular, g has a finite fixed point set. Since $Mod(S_2)$ is in the centralizer of g, $Mod(S_2)$ has an invariant measure, and hence, a global fixed point.

Again, apply the Thurston Stability Lemma to deduce that $Mod(S_2)$ acts trivially.

$Aut(F_n)$ and $Out(F_n)$

It is known that $Aut(F_n)$ and $Out(F_n)$ have trivial cohomology for $n \geqslant 3$.

So the above argument can be made to work for these groups also.

Open questions

Does every finite index subgroup of $Mod(S)$ have a subgroup with trivial cohomology?

Does every finite index subgroup of $Aut(F_n)$ or $Out(F_n)$ have a subgroup with trivial cohomology?