Non-convexity of Lyapunov Spectra

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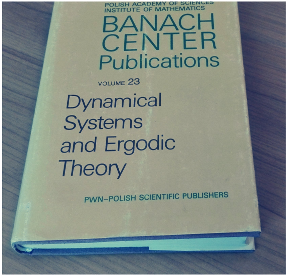
Warsaw (1986) and Cambridge (2000)



Tolya and I both attended a programme at the Banach center in 1986.
In 2000, we co-organized a 6 month programme at the Newton Institute, in Cambridge, UK.

Warwick (1980)

Tolya appears in the group photograph for the 1980 symposium at Warwick on Smooth Ergodic theory.

Lyapunov analysis : A general comment

Let $T : X \to X$ be a hyperbolic map (e,g, a diffeomorphism or an expanding map).

We can ask about the size of the set of points $x \in X$ for which the Lyapunov exponent exists takes a given value α, say, i.e.,

$$\lim_{n \to +\infty} \frac{1}{n} \log \|DT^n(x)\| = \alpha.$$

Question

What is the size of the set of points whose Lyapunov exponent is α?

As a starting point we can consider the (perhaps) more familiar case of a classical example of *multifractal analysis*.

But first we want to fix a particularly simple class of maps T.

Setting: Cookie cutters

Here X is a Cantor set in the unit interval and $T : X \to X$ is a (piecewise) linear expanding map.

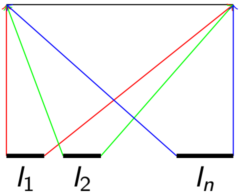


Figure: A cookie cutter

More precisely, let $I_1, \cdots, I_n \subset [0,1]$ be closed pairwise disjoint subintervals and consider a piecewise linear expanding map

$$T : \cup_{i=1}^{n} I_i \to [0,1]$$
$$T : x \mapsto a_i x + b_i \text{ for } x \in I_i$$

where $a_i = 1/|I_i| > 1$ and $b_i \in [0,1]$.

We then define $X = \cap_{n=0}^{\infty} T^{-n}[0,1]$ and let $T : X \to X$ be the restriction of the map defined above.

Simple(st) example of a cookie cutter: Middle $\frac{1}{3}$ Cantor set

In the special case of the usual middle $\frac{1}{3}$ Cantor set

$$X = \left\{ x = \sum_{n=1}^{\infty} \epsilon_n 3^{-n} : \epsilon_n \in \{0, 2\}, n \in \mathbb{N} \right\}$$

we can let $I_1 = [0, \frac{1}{3}]$ and $I_2 = [\frac{2}{3}, 1]$.

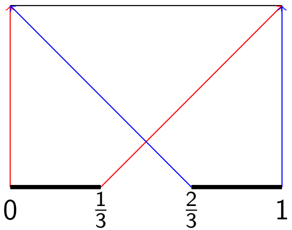


Figure: 2-interval cookie cutter

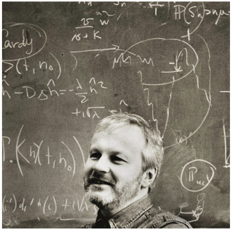
Consider the piecewise linear expanding maps $T_1 : I_1 \to [0, 1]$ and $T_2 : I_2 \to [0, 1]$ of the form

$$T_1(x) = 3x \quad \text{for } x \in I_1$$
$$T_2(x) = 3x - 2 \quad \text{for } x \in I_2$$

Then $X = \cap_{n=0}^{\infty} T^{-n}[0, 1]$ and let $T : X \to X$ and $T(x) = 3x \pmod 1$.

What is in a name?

The name "cookie cutter" was popularized by Dennis Sullivan.



The first reference (in Mathscinet) appears in the article of Bohr and Rand.

"'For clarity we are not going to treat the general case here, but instead concentrate on the proto- typical example of Sullivan's 'cookie-cutter' Cantor sets. In this way we avoid non-essential technical problems and quickly get down to the heart of the problem."

- T. Bohr and D. Rand, Physica D, 1987

A simple example in Multifractal Analysis

Consider (again) the middle $\frac{1}{3}$ Cantor set

$$X = \left\{ x = \sum_{n=1}^{\infty} \epsilon_n 3^{-n} : \epsilon_n \in \{0, 2\}, n \in \mathbb{N} \right\}.$$

Let

- $\mu_p = (p, 1-p)^{\mathbb{N}}$ be a Bernoulli measure on X.
- $T : X \to X$ be the natural (trebling) map defined by $T(x) = 3x \pmod 1$.
- Let $f : X \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in X \cap \left[0, \frac{1}{3}\right] \cap X \\ 0 \text{ if } x \in X \cap \left[\frac{2}{3}, 1\right] \cap X \end{cases}$$

Birkhoff Ergodic Theorem

In particular, the Ceséro (or Birkhoff) averages

$$A_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

are merely the proportion of the digit 0 in the first n terms $\epsilon_1(x), \epsilon_2(x), \cdots, \epsilon_n(x)$ in the base 3 expansion.

Theorem (Birkhoff Ergodic Theorem, 1931)

For the ergodic probability measure μ_p and a.e. (μ_p) the limit $\lim_{n\to+\infty} A_n f(x)$ exists and equals p.

Of course, different choices $0 < p < 1$ show there are at least some points (in fact a full μ_p measure set) with that frequency of 0s.

Size of sets with a given limit

Question

Given $0 < p < 1$, how many points x actually have that limit?

We can consider the size of the sets in terms of the Hausdorff Dimension.

Definition

For $0 < p < 1$ we define

$$X_p = \left\{ x \in X : \lim_{n \to +\infty} A_n f(x) = p \right\}$$

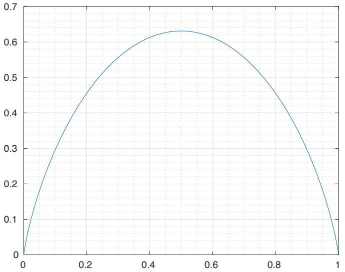
i.e., the set of points whose frequency of 0s in its expansion is p.
and $\mathcal{F}(p) = \dim(X_p)$.

Besicovich Multifractal result

In 1935 Besicovich considered the fequency of digits for diadic expansions in the unit interval. However, this easily adapts to show the following.

Lemma (After Besicovich)

$\mathcal{F}(p) := \dim(X_p) = \frac{-p \log p - (1-p)\log(1-p)}{\log 3}$.



Moving onto the Lyapunov spectrum

Let $T : X \mapsto X$ be a cookie cutter and consider next $f(x) = \log|T'(x)|$. In particular, in place of ergodic averages we can consider Lyapunov exponents, i.e.,

$$L_n(x) := \frac{1}{n} \log|(T^n)'(x)| \, (= A_n f(x)) \, ,$$

for $x \in X$ and $n \geq 1$, by the chain rule.

Definition

For $\alpha \in \mathbb{R}$ we define

$$Y_\alpha = \left\{ x \in X : \lim_{n \to +\infty} L_n f(x) = \alpha \right\}$$

and $\mathcal{L}(\alpha) = \dim(Y_\alpha)$.

On might anticipate that the Lyapunov spectrum (i.e., the function $\alpha \mapsto \mathcal{L}(\alpha)$) has similar properties to the Multifractal spectrum... or perhaps not.

Properties of the Lyapunov spectrum

Motivated by properties of the multifractal spectra $\mathcal{F}(\alpha)$ we can ask:

- Is $\alpha \mapsto \mathcal{L}(\alpha)$ analytic?
- Is $\alpha \mapsto \mathcal{L}(\alpha)$ convex?

Howie Weiss resolved the first question.

Theorem (H. Weiss)

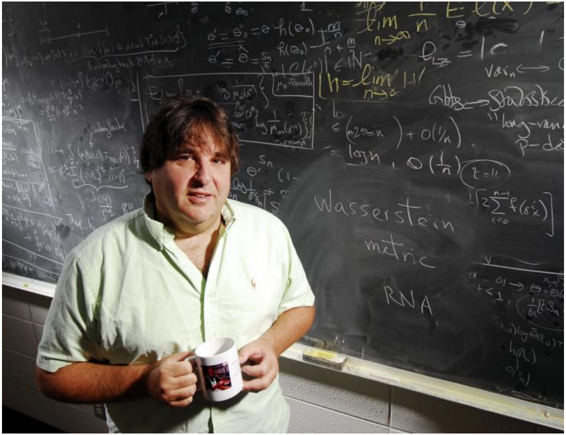
The Lyapunov spectrum $\mathcal{L}(\alpha)$ is real analytic.

Convexity of $\mathcal{L}(\alpha)$ was also (incorrectly) claimed by Howie Weiss:

"An important tool in studying these questions is the Lyapunov spectrum, which records the the Hausdorff dimension of the level sets for the apectum Lyapunov exponent. We show that for most conformal repellers, this []map] is real analytic and strictly convex."

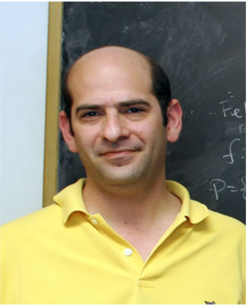
- H. Weiss, J. Stat. Phys., 1999.

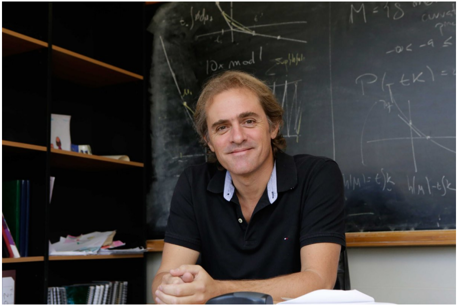
Howie Weiss



J. Kiwi and G. Iommi

However, Kiwi and Iommi came up with explicit counter examples.





There examples are based on cookie cutters with two branches with different expansion rates $b > a > 1$, say.

The Kiwi-Iommi counter examples

More precisely, let

- $b > a > 1$ with $\frac{1}{a} + \frac{1}{b} < 1$,
- $I_1 = [0, \frac{1}{a}]$ and $I_2 = [1 - \frac{1}{b}, 1]$,
- $T : I_1 \cup I_2 \to [0, 1]$ is defined by

$$T(x) = \begin{cases} ax & \text{if } x \in I_1 \\ bx + (1-b) & \text{if } x \in I_2. \end{cases}$$

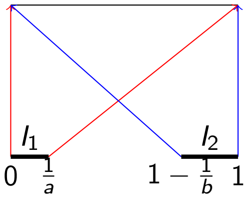


Figure: A cookie cutter

Then the Lyapunov spectrum has a simple explicit closed form expression

$$\mathcal{L}(\alpha) = \frac{1}{\alpha}\left(-\left(\frac{\log b - \alpha}{\log(b/a)}\right)\log\left(\frac{\log b - \alpha}{\log(b/a)}\right) - \left(\frac{\alpha - \log a}{\log(b/a)}\right)\log\left(\frac{\alpha - \log a}{\log(b/a)}\right)\right.$$

The lack of convexity in Kiwi-Iommi examples

In particular, they concluded the following.

Theorem (Iommi-Kiwi)

The function $\mathcal{L} : (\log a, \log b) \to \mathbb{R}$ *is convex if and only if*

$$\frac{\log b}{\log a} \leq \frac{\sqrt{2 \log 2} + 1}{\sqrt{2 \log 2} - 1} = 12.2733 \cdots$$

Thus for choices of a and b which do **not** satisfy this inequality there are inflection points.

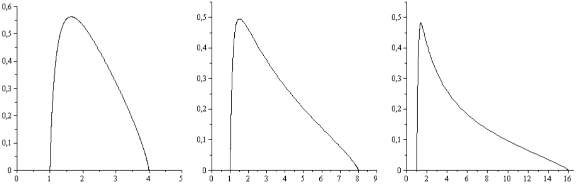


Figure: Fix $a = e^1$ and then (i) $b = e^1$; (ii) $b = 12.2733 \cdots$; and (iii) $b = e^{45}$

The Kiwi-Iommi's first (warm-up) problem

Iommi and Kiwi conjectured that their linear two branches examples cannot have have more than two points of inflection.

Conjecture (Iommi-Kiwi)

In this example can there be more points of inflection?

This conjecture is confimed in our first result.

Theorem (Jenkinson-P-Vytnova)

For two linear branches there are at most two points of inflection

(I am not sure what happens for nonlinear branches.)

The Kiwi-Iommi's second problem : More inflection points

Iommi and Kiwi also asked the natural question.

Conjecture (Iommi-Kiwi)

If we have a cookie cutter with more branches then can we have more points of inflection? Is the number of points of inflection bounded?

We have an answer for this too.

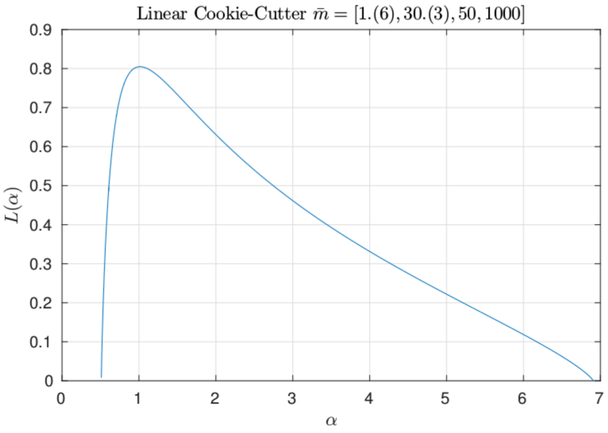
Theorem (Jenkinson-P-Vytnova)

For any $N > 0$ there are examples of cookie cutters T whose Lyapunov spectrum $\alpha \mapsto \mathcal{L}(\alpha)$ has at least N inflection points.

Before discussing the proof, let us look at some examples.

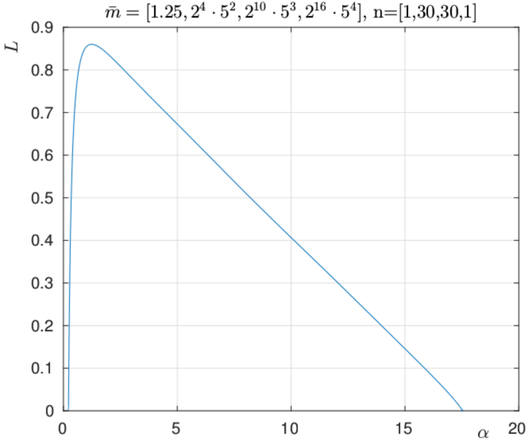
An ad hoc example

One can construct an explicit example (using 11 intervals) such that $\mathcal{L}(\alpha)$ has 4 points of inflection.



Another ad hoc example

One can construct an explicit example (using 62 intervals) such that $\mathcal{L}(\alpha)$ has 6 points of inflection.



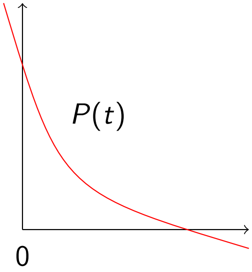
Pressure

The key to analyzing the Lyapunov spectrum $\mathcal{L}(\alpha)$ is the pressure function. Let $I_1, \cdots, I_n$ be the subintervals used in the definition of the cookie cutter.

Definition

The *pressure function* $P : \mathbb{R} \to \mathbb{R}$ takes the simple explicit form

$$P(t) = \log\left(\sum_{i=1}^{n} |I_i|^t\right).$$



The derivatives of Pressure

Recall that

$$P(t) = \log\left(\sum_{i=1}^{n} |I_i|^t\right).$$

Clearly (unless I made a mistake!) the first derivative is

$$P'(t) = \frac{\sum_{i=1}^{n} (\log |I_i|) |I_i|^t}{\sum_{i=1}^{n} |I_i|^t}$$

and the second derivative is

$$P''(t) = \frac{\sum_{i=1}^{n} (\log |I_i|)^2 |I_i|^t}{\sum_{i=1}^{n} |I_i|^t} - \left(\frac{\sum_{i=1}^{n} (\log |I_i|) |I_i|^t}{\sum_{i=1}^{n} |I_i|^t}\right)^2.$$

The Lyapunov spectra $\mathcal{L}(\alpha)$ can now be written in terms of the pressure $P(t)$ and $P'(t)$ ...

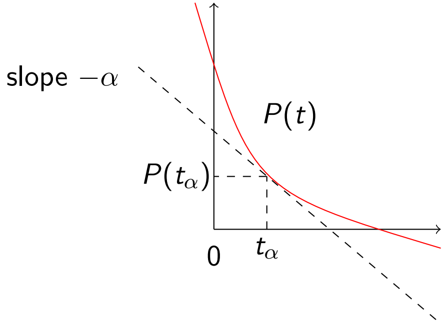
Pressure $P(t)$ and the Lyapunov spectrum $\mathcal{L}(\alpha)$

Lemma (H.Weiss)

Given $\alpha_- < \alpha < \alpha_+$ we can write

$$\mathcal{L}(\alpha) = \frac{1}{\alpha}(P(t_\alpha) + t_\alpha \alpha)$$

where t_α is the unique solution to $P'(t_\alpha) = -\alpha$.



Pressure and inflection points in $\mathcal{L}(\alpha)$

Moreover, the inflection points of $\mathcal{L}(\alpha)$ are characterized by the pressure.

Lemma (after Kiwi-Iommi)

The Lyapunov spectrum has a point of inflection at α_0 corresponding to t_0 (i.e., $\mathcal{L}''(\alpha_0) = 0$ and $P'(t_0) = -\alpha_0$) if and only if

$$2\frac{P''(t_0)P(t_0)}{P'(t_0)^2} = 1. \tag{1}$$

This gives us the following criterion.

Criterion for several inflection points

To find an example with Lyapunov spectra $\mathcal{L}(\alpha)$ with N inflection points it suffices to show that there are N solutions t_0 to the pressure equation (1).

An example revisited

Let us reconsider an earlier example (with 62 intervals):

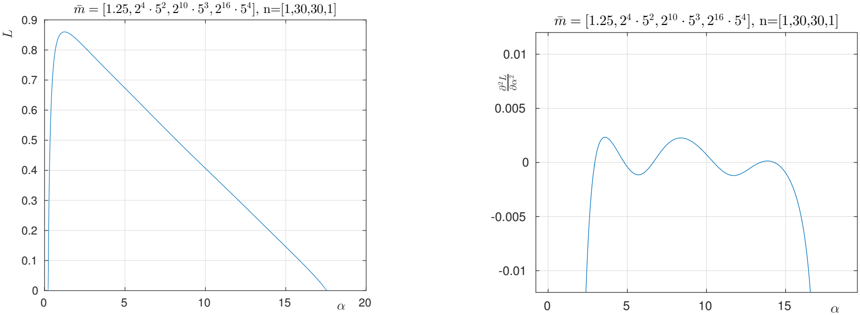


Figure: (i) The plot of $\mathcal{L}(\alpha)$; (ii) The plot of $2\frac{P''(t)P(t)}{P'(t)^2} - 1$

The number of inflection points is 4 (i.e., the number of zeros $t = t_0$ for $2\frac{P''(t)P(t)}{P'(t)^2} - 1$).

An infinite series

Consider the explicit function $F : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$F(t) = \sum_{j=1}^{\infty} 2^{j^2 - 2^j t}$$

We can formally define $\mathcal{P} : \mathbb{R}^+ \to \mathbb{R}$ by $\mathcal{P}(t) := \log F(t)$.

Strategy

Assume that we can show that there are **infinitely** many solutions $t = t_0$ to

$$2\frac{\mathcal{P}''(t)\mathcal{P}(t)}{\mathcal{P}'(t)^2} = 1$$

Then by truncating the series to $P(t) := \log \left(\sum_{j=1}^{M} 2^{j^2 - 2^j t} \right)$, say we still get examples with arbitrarily many solutions (for M sufficiently large), since $2^{j^2 - 2^j t} = 2^{j^2} 2^{-2^j t}$ this corresponds to the pressure function for a (finite branch) cookie cutter (2^{-2^j} = interval lengths, 2^{j^2} = multiplicity).

Solutions for the infinite series

It remains to show that for $\mathcal{P}(t) := \log \left(\sum_{j=1}^{\infty} 2^{j^2 - 2^j t} \right)$ there are **infinitely** many solutions $t = t_0$ to

$$2\frac{\mathcal{P}''(t)\mathcal{P}(t)}{\mathcal{P}'(t)^2} = 1$$

The method follows a suggestion of Victor Kleptsyn.

We want to choose two sequences $t_1 > m_1 > t_2 > m_2 > t_3 > \cdots$ such that

$$2\frac{\mathcal{P}''(t_i)\mathcal{P}(t_i)}{\mathcal{P}'(t_i)^2} > 1 \text{ and } 2\frac{\mathcal{P}''(m_i)\mathcal{P}(m_i)}{\mathcal{P}'(m_i)^2} < 1$$

and then apply the Intermediate Value Theorem. It suffices to take

$$t_j := \frac{2j+1}{2^j} \text{ and } m_j := \frac{t_j + t_{j-1}}{2} = \frac{6j-1}{2^{j+1}},$$

and to do pages and pages of calculus!

Finally ...

Thank you for your attention

Where limits don't converge

Question

But what about the set of points where the limit doesn't exist?

These form a set of zero measure (with respect to any ergodic measure μ) but the Birkhoff ergodic theorem. However, ...

Theorem (Barreira, Schmeling)

The set of points for which the averages don't converge have full dimension, i.e.,

$$\dim\left(\left\{x \in X : \lim_{n\to+\infty} A_n f(x) \text{ doesn't exist}\right\}\right) = \dim(X).$$