Dynamical properties of generalized pinwheel tilings

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2020 Vision for Dynamics
In memory of Anatole Katok
Bedlewo, Poland

August 11-16, 2019

Section 1

1-dimensional substitutions and tilings

- $\mathcal{A} = \{1, 2 \ldots, r\}$=alphabet, $r \geq 2$.
- $\mathcal{A}^*$= finite words on $\mathcal{A}$.
- $S : \mathcal{A} \to \mathcal{A}^*$ "substitution": $Sa = a_1 a_2 \ldots a_{e_a}$.
- $S^n : \mathcal{A} \to \mathcal{A}^*$ iterated substitution ($S : \mathcal{A}^* \to \mathcal{A}^*$).

Assume primitive: for all $a, b \in \mathcal{A}$ there exist n and k so that $b = (S^n a)_k$.

Define the substitution subshift by

$$X = \{x : x[j, j + \ell] = (S^n 0)[k, k + \ell] \subseteq \mathcal{A}^{\mathbb{Z}},$$

with $x = \ldots x_{-2} x_{-1}.x_0 x_1 x_2 \ldots$, and the left-shit map T.

One has $S : X \to X$. We usually assume S is "recognizable" (essentially, S is 1:1 on X).

The Perron-Frobenius suspension

$M = (m_{a,b})$ = the $r \times r$ incidence matrix x:

$$m_{a,b} := \#\{k : (Sb)_k = a\}.$$

Primitive implies $M^n > 0$.

Find the left and right Perron-Frobenius eigenvalue-eigenvectors

$$M\boldsymbol{r} = \lambda \boldsymbol{r} \qquad M^t \boldsymbol{\ell} = \lambda \boldsymbol{\ell}.$$

Note $\boldsymbol{\ell} > 0$, $\boldsymbol{r} > 0$ and $\lambda > 0$.
Usually we normalize $\boldsymbol{r} \cdot \mathbf{1} = 1$ and $\boldsymbol{\ell} \cdot \boldsymbol{r} = 1$.

Note that $\boldsymbol{r}$ defiens the frequencies of symbols, and ultimately determines the unique T-invariant measure on X. (T on X is minimal & uniquely ergodic).

Define $h : X \to \mathbb{R}_{\geq 0}$ by $h(x) := \boldsymbol{\ell}_{a_0}$ and construct the corresponding suspension:

$$\widetilde{X} = \{(x, r) : x \in X, r \in [0, h(x))\}.$$

with suspension flow H^s, $s \in \mathbb{R}$.

Orbits of H^s in $\widetilde{X}$ naturally tiled by intervals

$$\widetilde{\mathcal{A}} = \{I_a = [0, \boldsymbol{\ell}_a] : a \in \mathcal{A}\}.$$

In particular, the tiling of $\mathbb{R}$ by the intervals $\widetilde{\mathcal{A}}$ are $\tilde{x} \sim (x, r) \in \widetilde{X}$:

$$\tilde{x} = \{\dots I_{a_{-2}} I_{a_{-1}} \cdot_r I_{a_0} I_{a_1} I_{a_2} \dots\}$$

where $x = \dots a_{-2}a_{-1}.a_0a_1, \dots$ and $r \in [0, \boldsymbol{\ell}_{a_0})$.

Define the tiling substitution $G(I_a) := I_{a_1} I_{a_2} \ldots I_{a_{e_a}}$ where $Sa = a_1 a_2 \ldots a_{e_a}$. Use to define a tiling space $\widetilde{X}$ with tiling topology (i.e., $d(\tilde{x}, \tilde{y}) \leq \epsilon$ if $\tilde{x}$ and $\tilde{y}$ agree prefectly on $(-1/\epsilon, 1/\epsilon)$ after an ϵ shift).

- H^s acts on $\widetilde{X}$ by translation.
- G can be used to define $\widetilde{X}$ directly.
- $\tilde{X}$ is a tiling space. Compact metric in "tiling topology".
- Tiling topology on $\widetilde{X}$ is same as the "product topology".
- Always strictly ergodic. Unique invariant measure comes from right eigenvector $\boldsymbol{r}$.

Extend G to a homeomorphism $G : \widetilde{X} \to \widetilde{X}$.

- This expands a tiling by λ and substitutes the elongated tiles.

It is hyperbolic: a "Smale space" in the terminology of Putnam:

- Two tilings that differ by a translation move apart.
- Two tilings that agree in a neighborhood of 0 move together.

The partition $\xi = \{\xi_a = \{(x, s) : x_0 = a\}$ is a Markov partition.

There is a commutation relation

$$GH^s = H^{\lambda s}G.$$

Section 2

Some examples and results

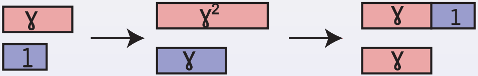
Fibonacci substitution

Substitution S: $\quad a \to b \quad b \to ba.$

Iterate $b \to ba \to bab \to babba \to babbabaa \ldots$
Substitution shift $X = \{\ldots ba.babba \ldots\} \subseteq \{a, b\}^{\mathbb{Z}}$, with shift T.

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ \gamma \end{pmatrix}, \quad \gamma = \frac{1+\sqrt{5}}{2} \sim 1.6180.$$

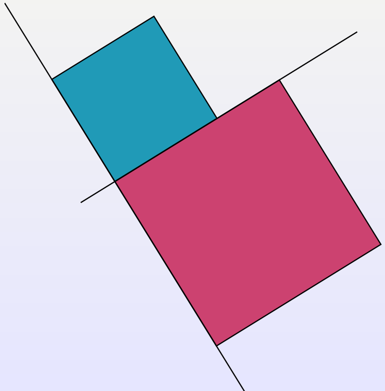
Tiles: $I_a = [0, 1]$, $I_b = [0, \gamma]$. Tiling substitution G:
$I_a \to I_b, I_b \to I_b I_a$, expansion γ.



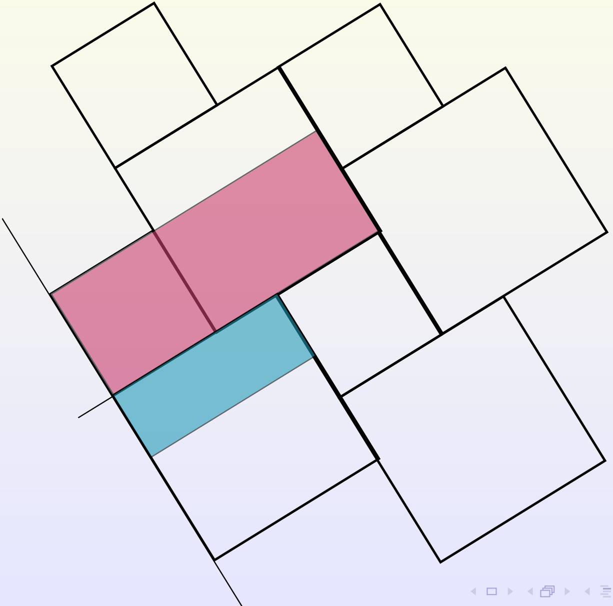
Tiling space $\widetilde{X} = \{\ldots I_b I_a \cdot_s I_b I_a I_b I_b I_a \ldots\} \subseteq \{I_a, I_b\}^{\mathbb{R}}$ with translation flow H^s.

Strictly ergodic, entropy 0 (linear complexity), pure point spectrum: $\mathbb{Z}[\gamma]$.

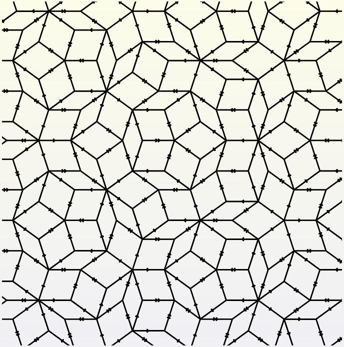
Suspension and Markov partition

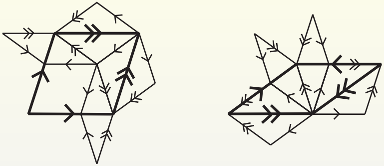


Markov partition



Penrose tilings





Expansion $\lambda = \frac{1+\sqrt{5}}{2}$ tiling substitution G. Translation flow $H^{\lambda \boldsymbol{s}}$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Penrose tiling dynamical system has pure point spectrum: $\mathbb{Z}\left(e^{2\pi i/5}\right) \subseteq \mathbb{C} \sim \mathbb{R}^2$. Satisfies commutation relation: $GH^{\boldsymbol{s}} = H^{\lambda \boldsymbol{s}} G$.

Theorem

Assume a substitution S (or tiling substitution G with finitely many prototiles) is primitive and recognizable. Then (X, T) (or $(\widetilde{X}, H^s)$):

- *is minimal and uniquely ergodic,*
- *may have discrete, mixed, or continuous spectrum (i.e., may be weakly mixing).*
- *has all eigenfunctions continuous.*
- *never strongly mixing but may be topological mixing.*
- *may have some absolutely continuous spectrum, but no pure Lebesgue spectrum,*
- *Pure singular continuous spectrum is possible*.*
- *Always finite spectral multiplicity and entropy zero.*

Discrete substitutions:

- Minimality for discrete substitutions goes back (at least) to Gottschalk (1969), and unique ergodicity to Kamae (1969) and Host (1986).
- Host (1986) also proved continuity of eigenfunctions and a condition for their existence (or not) involving Pisot numbers.
- Entropy zero and finite spectral multiplicity.
- No mixing and possibility of topological mixing example due to Dekking-Keane (1976). Weak mixing $\implies$ topological mixing (2-letter) Kenyon-Sadun-Solomyak, (2005).

Most results generalized to $\mathbb{R}$ (and many to $\mathbb{R}^d$) Solomyak (1996). Topological mixing for 2-tile weakly mixing substitution tiling flows due to Kenyon-Sadun-Solomyak, (2005).

Results for similar dynamical systems

- Interval exchange transformations
  - Generically minimal (Keane, 1975), uniquely ergodic (Veech 1978, Masur 1982) and weakly mixing (Avila-Forni, 2004).
  - Never strongly mixing (Katok, 1980) but generically topological mixing (Chaika, 2011; Chaika-Fickenscher, 2013). Partial mixing? (Chaika).
  - Finite spectral multiplicity: $m \leq i - 1$ (Oseledec, 1966).
  - Entropy zero.

- Rank 1 $\mathbb{Z}$-actions and $\mathbb{R}$-actions.
  - Ergodic (uniquely), simple spectrum $(m = 1)$. Entropy zero. Sometimes minimal.
  - Can be weakly (chacon, 1967) or strongly mixing (Ornstein, 1974) $\implies$ mixing of all orders (Kalikow, 1984; Rhyzakhov, 1993)
  - In $\mathbb{Z}$, continuous spectrum always singular.

Also: finite rank, $\mathbb{Z}^d$ or $\mathbb{R}^d$. "Fusion": Frank-Sadun (2015).

Thue-Morse substitution:

$a \to ab$
$b \to ba.$

Has point spectrum $\mathbb{Z}[1/2]$, but also a singular continuous complementary component. Simple spectrum.

Rudin-Shapiro substitution:

$a \to ab \quad b \to ac$
$c \to db \quad d \to dc.$

Also has point spectrum $\mathbb{Z}[1/2]$, but complementary component absolutely continuous. Non-simple spectrum: $m = 2$.

Baake and Grimm have an example with mixed spectrum of all three types.

Substitution $a \to b$, $b \to abbb$. "Non-Pisot":

$$\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}^t \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \lambda = \frac{1+\sqrt{13}}{2} \sim 2.3028, \lambda', \sim -1.3028$$

Tiling substitution $\mathcal{S}$, expansion λ

- Weakly mixing, not strongly mixing (Solomyak, 1997), but topologically mixing (Solomyak, Kenyon, Sadun, 2005).

Theorem (Baake, Frank, Grimm, R (2019))

Has purely singular continuous diffraction spectrum.

Related examples: Baake, Grimm, Gahler, Manibo (2019).

Let Λ_x be the set of endpoints of a tiling $x \in \widetilde{X}$ and let f be a function on $\widetilde{X}$ that is a "bump" on each $y \in \Lambda_x$. The diffraction spectrum is the (finite Borel) measure $\Sigma_{\widetilde{X}} = \sigma_f$ on $\mathbb{R}$. In particular, it has Fourier transform

$$\widehat{\Sigma}_{\widetilde{X}} := \widehat{\sigma_f}(s) := \int_{\mathbb{R}} e^{2\pi i s t} d\sigma_f(t) =< f \circ H^s, f > .$$

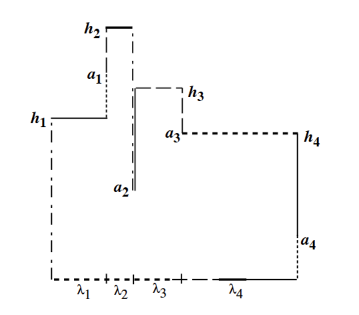
- If H^s has pure point spectrum then $\Sigma_{\widetilde{X}} = \sigma_H$ (f has maximal spectral type in this case).
- Otherwise, it is possible that $\Sigma_{\widetilde{X}} << \sigma_H$ (f does not have maximal spectral type).
- Cases are known where inequality is strict.

Four interval exchange. T. Fitzkee, 2003:

$1 \to 1424, 2 \to 142424, 3 \to 14334, 4 \to 1434$

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix}$$

$\lambda =\sim 4.39026$, $\boldsymbol{\ell} \sim (1.09529, 1.71333, 1.29496, 1)$.



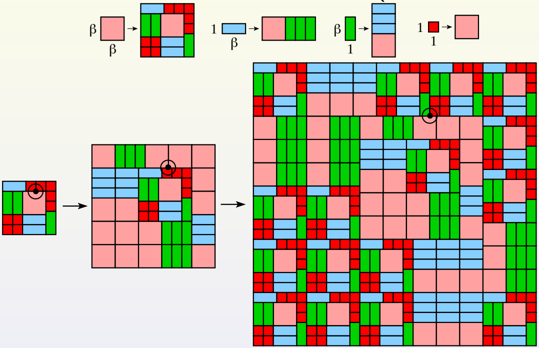
Weakly mixing flow H^s along stable leaves of pseudo-Anosov map G (up to almost 1:1 extension).

Section 3

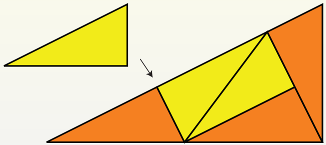
Infinite local complexity (ILC)

Infinite local complexity (ILC) example

"Product variation" of $a \to abbb$, $b \to a$ (N. Frank-R, 2007).



- infinitely many 2-tile patches (Frank-R, 2007): dot in pink tile moves to infinitelly many places
- ILC tiling systems with finitely many prototiles (like this) have essentially same theory as FLC case (Lee-Solomyak 2018)
- Singular diffraction (Baake-Grimm, 2018).

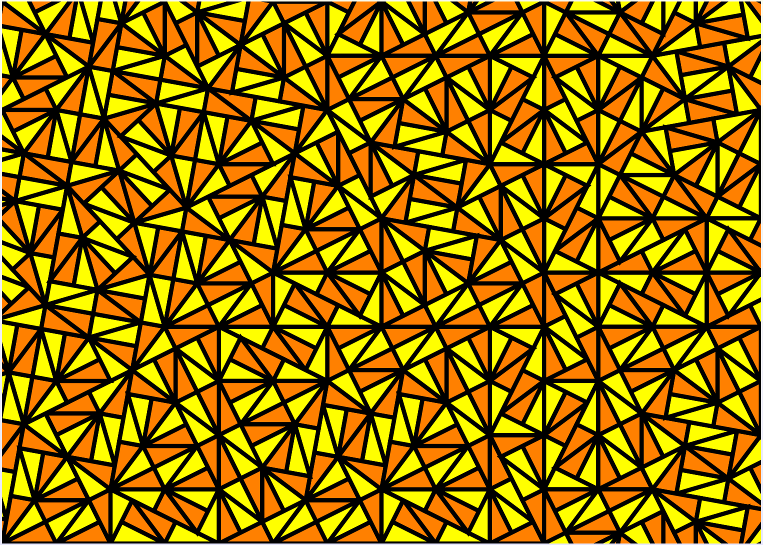


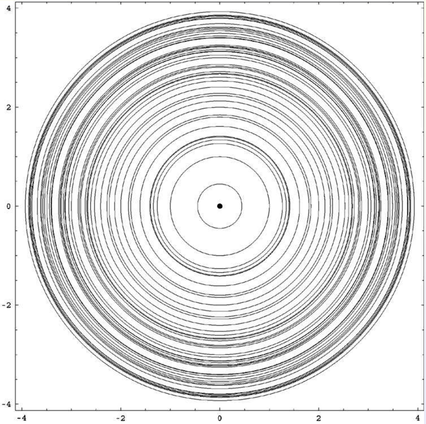
Conway-Radin "pinwheel" substitution $\mathcal{S}$: $\theta = \arctan(1/2)$.
Infinite local complexity due to infinitely many tiles (up to rotation): tiling space X is rotation invariant.

Weakly mixing (Radin, 1994). Proof: Spectrum rotation invariant, but discrete spectrum countable but also rotation invariant.

Radin conjecture: mixing and pure Lebesgue spectrum unresolved, but numerical evidence against it.

Pinwheel tiling

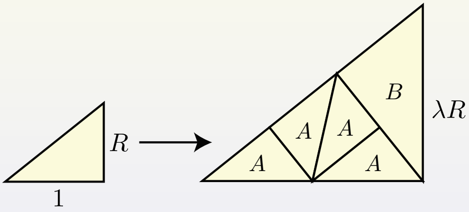




Moody, Postkinov, Strungaru, 2006.

Sadun Generalized Pinwheel

Fix $0 < R \leq 1$ and expansion $\lambda = \max\{\sin\theta, (1/2)\cos\theta\}^{-1}$.



With appropriate choice of R, the tiling space $\widetilde{X}$ has infinitely many scales and rotations.

We will show this action is mixing, multiple mixing and has Lebesgue spectrum.

Section 4

1-dimensional VTL substitution

Fix $R > 1$. Consider the Hilbert cube

$$Q = [1, R+1]^{\mathbb{Z}} = \{x = \dots a_{-1}.a_0 a_1 a_2 \cdots : a \in [1, R+1]\}$$

with shift $(Tx)_k = x_{k+1}$.

Substitution: let $\lambda = \frac{R+1}{R}$. Define $S : [1, R+1] \to [1, R+1]^*$ by

$$a \to a_1 \text{ if } a \in [1, R)$$
$$a \to a a_2 \text{ if } a \in [R, R+1],$$

where $a_1 = \lambda a$ and $a_2 = (\lambda - 1)a$.

Make a Hilbert cube substitution subshift $X \subseteq [1, R+1]^{\mathbb{Z}}$.

Define the suspension space $\widetilde{X}$ where $h(x) = a_0$. The substitution tiling flow the suspension flow H^s $(s \in \mathbb{R})$ over X.

- The *prototiles* are $\mathcal{I} = \{I_a = [0, a] : a \in [1, R+1]\}$.
- Tiling substitution $G : \mathcal{I} \to \mathcal{I}^*$ is defined by

$$\begin{aligned} I_a &\to I_{a_1} \text{ if } a \in [1, R) \\ I_a &\to I_a I_{a_2} \text{ if } a \in [R, R+1], \end{aligned}$$

- Tilings $\tilde{x} = \{\dots I_{a_{-2}} I_{a_{-1}} ._r I_{a_0}, I_{a_1} \dots\}$ where $0 \leq r < a_0$. Note that $r =$ position of time 0 in I_{a_0}.
- H^s acts by translation.

Comment: slightly different notion of tiling topology needed.

The case $R = 2$ was the case studied by Frank and Sadun (2009).

Here $\lambda = \frac{R+1}{R} = \frac{3}{2}$, and $S : [1,3] \to [1,3]^*$ is given by

$$a \to \left(\tfrac{3}{2}a\right) \text{ if } a \in [1,2)$$
$$a \to (a)\left(\tfrac{1}{2}a\right) \text{ if } a \in [2,3].$$

Note: expansion by $\lambda = \frac{3}{2}$.

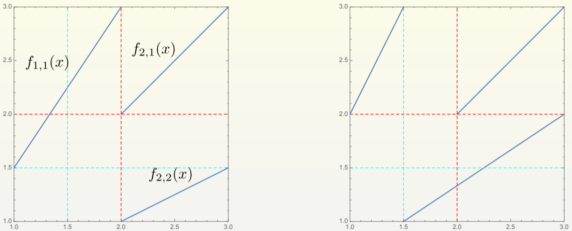
Theorem (Frank-Sadun, 2009)

(In the case $R = 2$) the tiling flow H^s is minimal, uniquely ergodic, entropy zero and has infinitely many tile lengths.*

Section 5

Unique ergodicity and mixing

The invariant measure



The "matrix" M and corresponding Perron-Frobenius operator M^* (in the case $R = 2$):

$$M(a) = \begin{cases} \{(3/2)a\} & \text{if } a \in [1,2), \\ \{a, (1/2)a\} & \text{if } a \in [2,3], \end{cases}$$

and for $\rho \in L^1([1, R+1], da)$:

$$(M^*\rho)(a) = \begin{cases} 2\rho(2a) & \text{if } a \in [1, 3/2), \\ (2/3)\rho((2/3)a) & \text{if } a \in [3/2, 2), \\ \rho(a) + (2/3)\rho((2/3)a) & \text{if } a \in [2,3]) \end{cases}$$

In the case $R = 2$:

$$\rho(a) = \begin{cases} \frac{1}{a^2} & \text{if } a \in [1,2) \\ \frac{3}{a^2} & \text{if } a \in [2,3] \end{cases} = \frac{\eta(a)}{a^2},$$

where $\eta_1(a)$ is a step function. It satisfies $M^*\rho = \lambda\rho$, so $\frac{M^*}{\lambda}\rho = \rho$ on $[1,3]$.

In general:

$$\rho(a) = \begin{cases} \frac{1}{a^2} & \text{if } a \in [1,R) \\ \frac{R+1}{a^2} & \text{if } a \in [R,R+1]. \end{cases}$$

A T-invariant probability measure μ on $[1, R+1]^{\mathbb{Z}}$ specified by consistent choice of probability measure μ_{2n+1} on each cylinder $[1, R+1]^{2n+1}$ (centered at 0).

For $n = 1$, we use $d\mu_1(a) = \rho(a)da = \eta_1(a)\frac{da}{a^2}$.

Use "supertiles" to extend this to each $[1, R+1]^{2n+1}$:

$$d\mu_n(\vec{a}) = \eta_n(\vec{a})\frac{da}{a^2},$$

where $\eta_n(\vec{a})$ is a step function, and

$$\vec{a} = (a_{-n}, \ldots, a_{-1}, a, a_1, \ldots a_n) \in [1, R+1]^{2n+1}.$$

- Define $\widetilde{\mu} = \mu \times dr$ on $\widetilde{X} = \{(x, r) : x \in X, r \in [0, h(x))\}$.

The substitution map G embeds in a flow G^t ($G = G^{\ln \frac{3}{2}}$).

- For G^t expand by e^t then subdivide (recursively, ratio $R : 1$) any tile longer than $R + 1$.

The measure μ is G^t invariant.

There is a commutation relation: $G^t \circ H^s = H^{se^t} \circ G^t$.

The flow G^t is hyperbolic:

- $W^s(x, r) = \{(y, r) : x_{[-N,N]} = y_{[-N,N]}, N \geq 0\}$.
- $W^u(x, r) = \{H^s(x, r) : s \in \mathbb{R}\}$.

LEMMA

Unique ergodicity follows from minimality (which follows from primitivity).

Essentially, we copy the proof of unique ergodicity of horocycle flow (e.g., Coudene, 2009).

THEOREM

The flow H^s has Lebesgue spectrum, therefore strongly mixing. In fact, mixing of all orders.

The commutation relation $G^t \circ H^s = H^{se^t} \circ G^t$ implies H^s is isomorphic to all its time changes.

LEMMA (KATOK-THOUVENOT, 2006)

A flow isomorphic to all its time hanges has Lebersgue spectrum.

No proof given.

The maximal spectral type satisfies $\sigma_H \sim (e^t)^* \sigma_H$ for all t. Lebesgue measure is Haar measure for $(\mathbb{R}^+, \cdot)$. However σ_H is finite, so can't have $\sigma_H = (e^t)^* \sigma_H$

THEOREM (MACKEY-WEIL THEOREM)

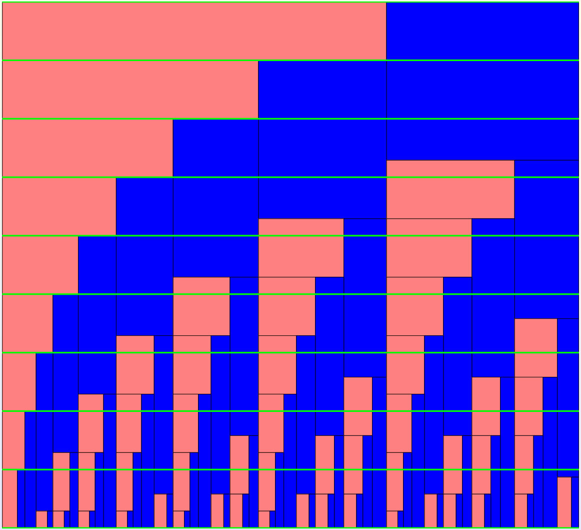
In a Polish group (eg, $(\mathbb{R}^+, \cdot)$) the only translation invariant measure class is the Harr measure class.

Comment: Horocycle flow has countable Lebesgue spectrum.
What is multiplicity for H^t?

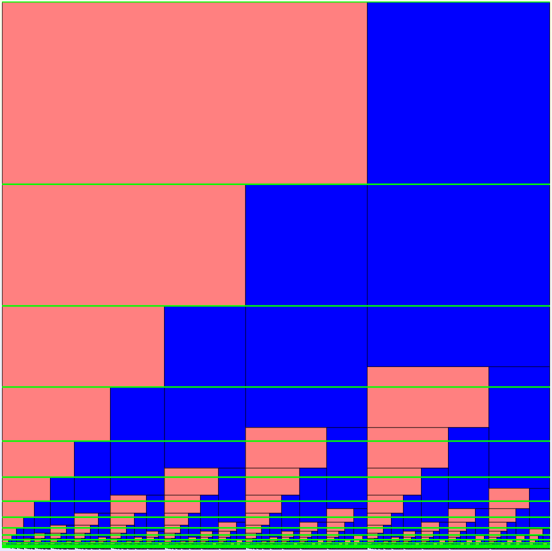
THEOREM (RHYZAKOV, 1991)

If a measure preserving flow satisfies H^s is isomorphic to H^1 for all $s > 0$ (or more generally, for a set of positive Lebesgue measure) then it is mixing of all orders.

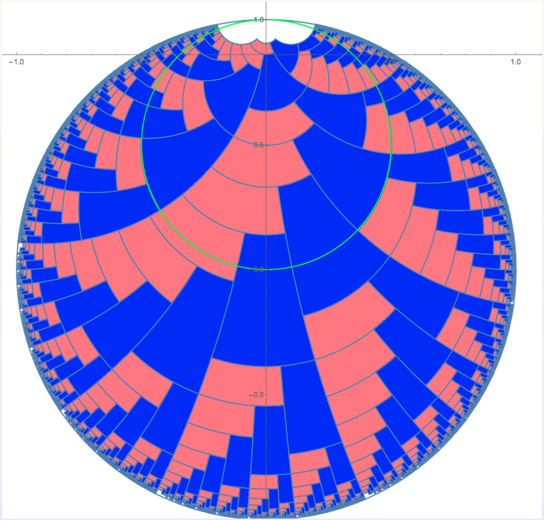
Tiling of the (s, t)-plane



Upper half-plane: (s, a) with $a = e^t$



In the disc model



Section 6

Primitivity

The map $F:[1,R+1]\to[1,R+1]$ ("part" of substitution) is defined

$$F(a)=\begin{cases}\frac{R+1}{R}a & \text{if } a\in[1,R)\\ \frac{1}{R}a & \text{if } a\in[R,R+1].\end{cases}$$

Define primitivity' to mean every a has dens M forward orbit.

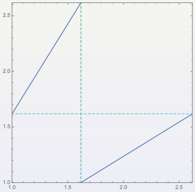


Figure shows the case $R=\frac{1+\sqrt{5}}{2}$.

LEMMA

Primitive implies minimal.

THEOREM

The tiling flow H^s is strictly ergodic if and only if g is primitive.

Apply $\phi : [1, R+1] \to [0,1]$, $\phi(a) = \log_{R+1}(a)$, to see F is conjugate to rotation on $[0,1]$ by $\alpha = \log_{R+1}(\frac{R+1}{R})$.

- "Primitivity" if and only if $\alpha \in \mathbb{Q}^c$.
- Otherwise, $\alpha = \frac{p}{q} = \log_{R+1}(\frac{R+1}{R})$ and solve for R:
$$q \log R = (q-p) \log(R+1),$$

or

$$R^q - (R+1)^{q-p} = 0.$$

Unique real root $R > 1$.

Example. $\alpha = \frac{1}{2}$ implies $R = \frac{1+\sqrt{5}}{2}$.

For $\alpha = \frac{p}{q}$ let $R = R_{p,q}$ denote the corresponding parameter.

Then for any $j \in [1, R+1]$, for all n, $S^n(j)$ has only q different values (tile lengths) the orbit of a rational rotation. In other words, the substitution is FLC!

Let $\mathcal{A} = \{F^n(j_0)\}_{n=0}^{q-1} = \{j_0, j_1, \ldots, j_{q-1}\}$. Then

$$\begin{aligned} j_a &\to j_{a+p} && \text{if } a < q-p, \\ j_a &\to j_a j_{a+p-q} && \text{if } a \geq q-p. \end{aligned}$$

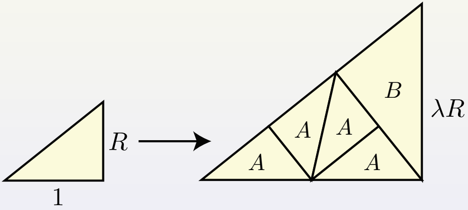
Example: The case $\alpha = 1/2$, $R = (1/2)(1+\sqrt{5})$ is the Fibonacci substitution: $j_0 \to j_1$, $j_1 \to j_1 j_0$ on $\mathcal{A} = \{j_0, j_1\}$. This is the only Pisot case.

Section 7

Sadun Pinwheel

Sadun Generalized Pinwheel

Fix $0 < R \leq 1$. The prototiles are (ℓ, r)-right-triangles with $r/\ell = R$ (angle $\theta = \tan^{-1}(R)$). Let $a = \frac{1}{2\sqrt{1+R^2}} = \frac{1}{2}\cos\theta$, $b = \frac{R}{\sqrt{1+R^2}} = \sin\theta$.



- Restrict $\ell \in [\zeta, 1]$ where $\zeta = \min\{a, b\}$.
- Expansion $\lambda = \max\{a, b\}^{-1}$.

Substitution: Expand $\ell \in [\zeta, 1]$ by λ. Subdivide if $\lambda\ell \geq 1$ (see later picture).

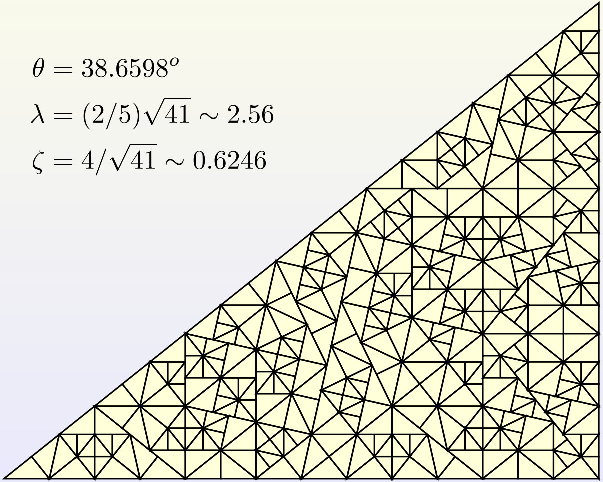
- A larger if $R < 1/2$; B larger if $R > 1/2$.

$R = 4/5 > 1/2$

$\theta = 38.6598^o$

$\lambda = (2/5)\sqrt{41} \sim 2.56$

$\zeta = 4/\sqrt{41} \sim 0.6246$

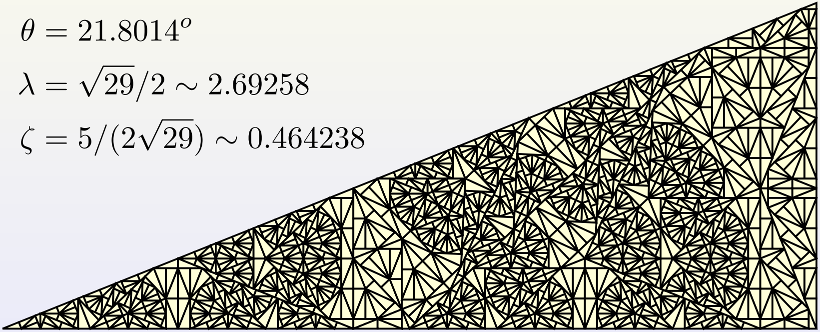


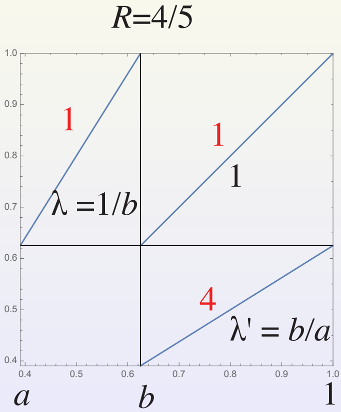
$R = 2/5 < 1/2$

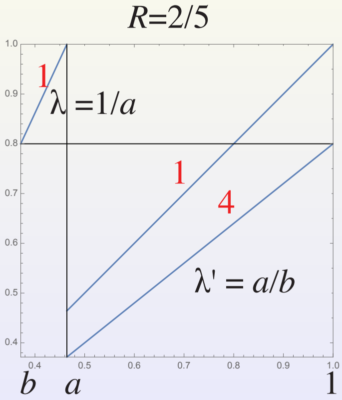
$\theta = 21.8014^o$

$\lambda = \sqrt{29}/2 \sim 2.69258$

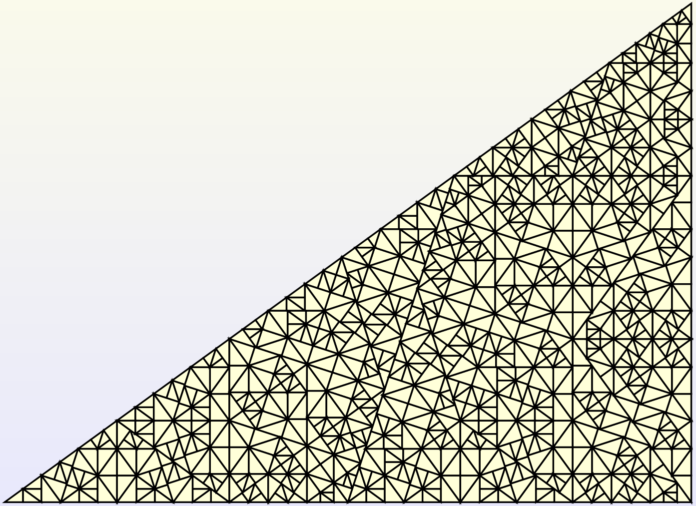
$\zeta = 5/(2\sqrt{29}) \sim 0.464238$




R=4/5
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1
1
1
λ =1/b
4
λ' = b/a
a
b
1


R=2/5
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1
λ =1/a
1
4
λ' = a/b
b
a
1

$\theta = 36^o$



"Finitary" vs "Generic"

Theorem (L. Sadun (1998))

- *Finitely many rotations* $\iff$ $\theta \in 2\pi\mathbb{Q}$.
- *Finitely many sizes* $\iff$ $\frac{\log(\sin\theta)}{\log((1/2)\cos\theta)} = \frac{\log R - \log\sqrt{1+R^2}}{-(\log 2 + \log\sqrt{1+R^2})} \in \mathbb{Q}$

Corollary

The only case with both finitely many sizes and finitely many rotations is $R = 1$ ($\theta = 45^o$). It has 3 sizes and 8 rotations.

Call cases of neither "generic".

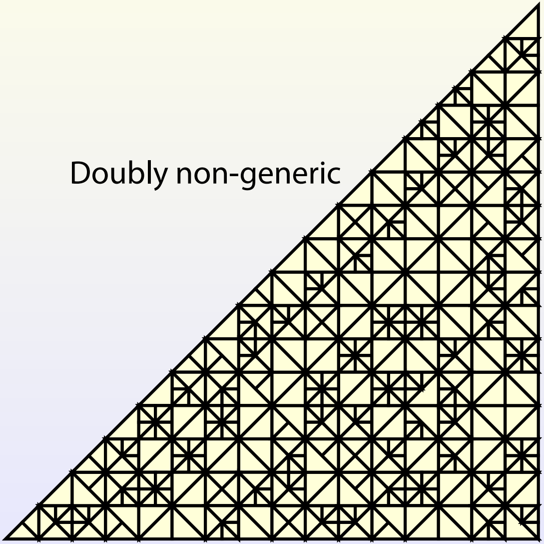
Theorem (Sadun, 1998)

In the generic cases, $H^{\vec{s}}$ is minimal, uniquely ergodic and weakly mixing.

Conjecture (Sadun, 1998): Lebesgue spectrum.

$\theta = 45^o$

Doubly non-generic



Theorem

In the generic cases, $H^{\vec{s}}$ has Lebesgue spectrum.

In these cases the translation action $H^{\vec{s}}$ and the expansion G^t satisfy

$$G^t \circ H^{\vec{s}} = H^{\vec{s}e^t} \circ G^t.$$

This, in itself, is not enough to guarantee Lebesgue spectrum because $\{\vec{s}e^t : t \in \mathbb{R}\}$ is 1-dimensional.

However, the tiling space also is R_θ invariant $\theta \in \mathbb{R}$.

It follows that the measure class σ_H is invariant under both rotation and dilation (a cylinder). This implies it is the class of Lebesgue measure.

Here is the multidimensional version of Ryzhikov's theorem, which does not quite work for our purposes.

Theorem (Ryzhivov, 1998)

Let $H^{\vec{s}}$ be a weakly mixing $\mathbb{R}^d$-action. Let $(H|_{\mathbb{Z}^d})^{\vec{n}}$ be its restriction to $\mathbb{Z}^d$ and let $(H|_{\mathcal{L}_{\vec{r}}})^{\vec{n}}$ be its restriction to $\mathcal{L}_{\vec{r}} := \mathbb{Z}[r_1\vec{e}_1, \ldots, r_d\vec{e}_d]$, $\vec{r} = (r_1, \ldots r_d) \in \mathbb{R}^d$. If there is a positive Lebesgue measure set of $\vec{r} \geq 0$ so that $H|_{\mathbb{Z}^d}$ is isomorphic to $H|_{\mathcal{L}_{\vec{r}}}$ then $H^{\vec{t}}$ is mixing of all orders.

Theorem in progress

In the generic cases, the Sadun Pinwheel tilings are $H^{\vec{s}}$ is mixing of all orders.

Think of $\vec{s} \in \mathbb{C}$. The rotation and expansion become complex multiplication by $\vec{s} = re^{i\theta} \in \mathbb{C}$. The one dimensional proof then works for $\mathbb{C} = \mathbb{R}^2$.