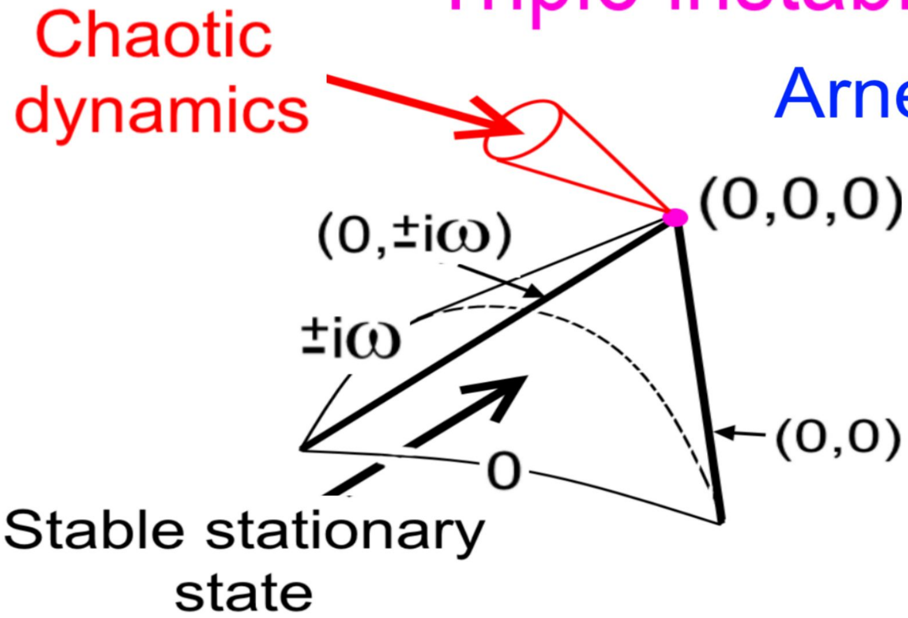
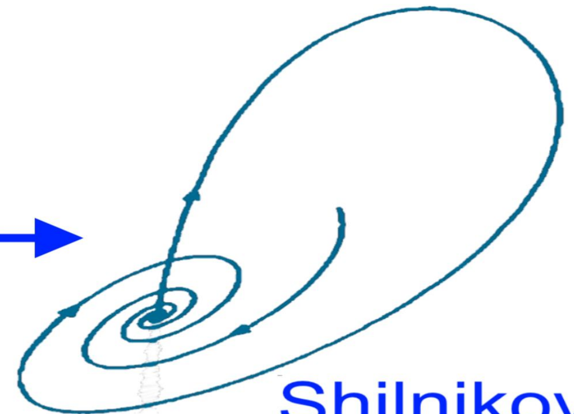
Triple instability as a source of Chaos



Arneodo, Coullet, Spiegel, Tresser (1985)
"Asymptotic Chaos" (Physica D):

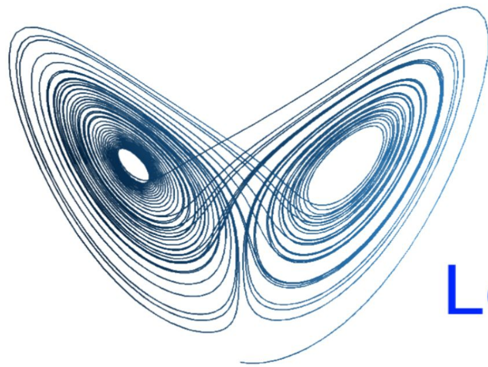
(0,0,0) →



Shilnikov chaos

Vladimirov, Volkov (1993)
Low intensity chaotic operations of a laser with a saturable absorber (Optics Communications):

(0,0,0) + symmetry →



Lorenz attractor

Three zero eigenvalues:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{du}{dt} = A\,u + O(u^2) \longleftrightarrow \begin{cases} \dfrac{dx}{dt} = y + O(u^2) \\ \dfrac{dy}{dt} = z + O(u^2) \\ \dfrac{dz}{dt} = O(u^2) \end{cases}$$

$$\dddot{x} = x^2 + q\,x\,\dot{x} + \ldots$$

Small perturbation:

$$\dddot{x} + \alpha\ddot{x} + \beta\dot{x} + \gamma x = x^2 + q\,x\,\dot{x} + \ldots$$

small parameters (α, β, γ)

Universal model for the triple instability:

$$x''' + a\,x'' + b\,x' + c\,x = x^2$$

not small

SCALING: time to τ

x to τ^3

$\alpha = a\tau$

$\beta = b\tau^2$

$\gamma = c\tau^3$

$$\dddot{x} + \alpha\ddot{x} + \beta\dot{x} + \gamma x = x^2 + q\,x\,\dot{x} + \ldots$$

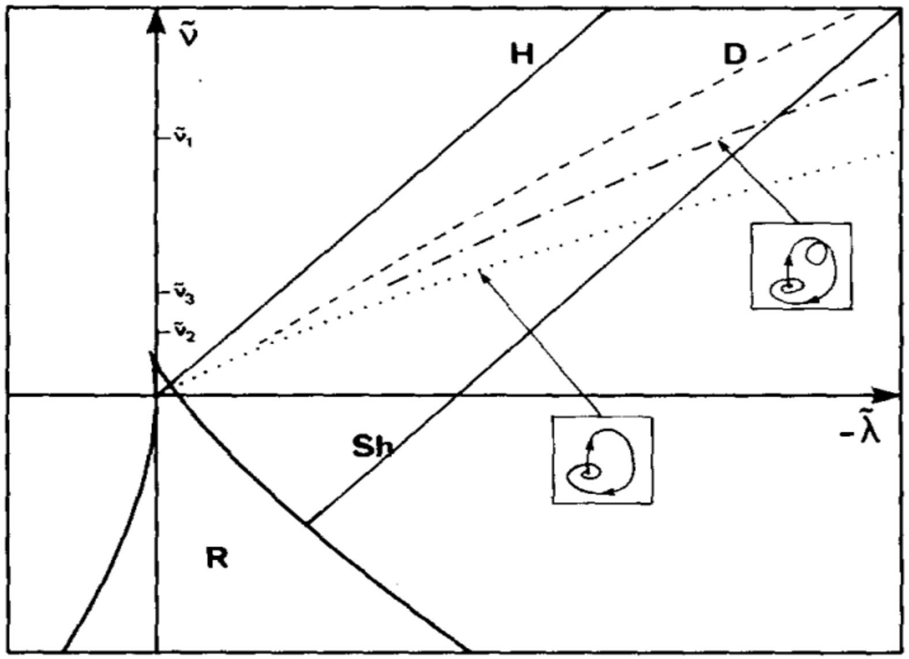
small parameters

$q\,x\,\dot{x} + \ldots = O(\tau)$

Triple instability:

$$x''' + a\,x'' + b\,x' + c\,x = x^2$$

there exists a region of (a,b,c)
for which the dynamics is chaotic

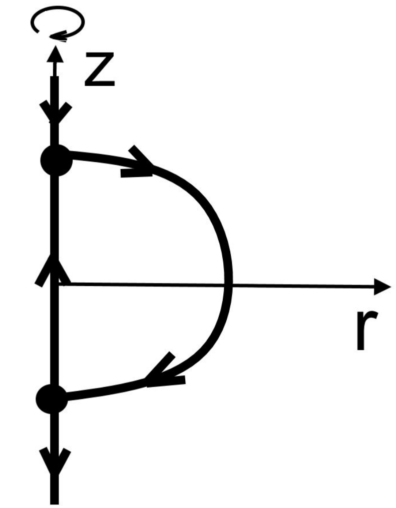


Triple instability: $x''' + a\,x'' + b\,x' + c\,x = x^2$

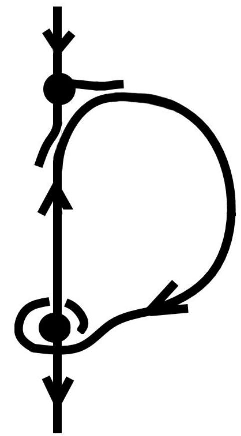
there exists a region of (a,b,c) for which the dynamics is chaotic

Proof. When $a=c=0$, $b=\omega^2$ the equilibrium at $x=0$ has characteristic exponents 0 and $\pm i\omega$. Its perturbation can be brought to the normal form

$$\dot{r} = 2rz + \dots$$
$$\dot{z} = \mu - z^2 - r^2 + \dots$$



With an additional effort - a saddle-focus loop (Ibanez, Rodriguez)



Triple instability of a periodic orbit

$$x''' + a\,x'' + b\,x' + c\,x = x^2 + \ldots$$

a small time-periodic perturbation

Triple instability leads to chaos

Q. Which type of chaos?

A. ALL types of chaos.

Because perturbing the triply unstable periodic point
one can create a periodic spot
and perturbing a periodic spot one can create
everything

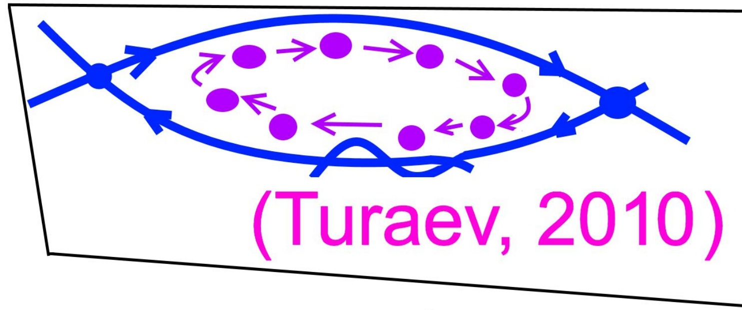
Periodic spot = an open ball filled by periodic orbits of the same period (so the return map is identity)

Theorem (Turaev, 2006; Berger, Gourmelon, 2019)
Given any orientation-preserving C^∞-diffeomorphism g of a closed unit ball into R^n, arbitrarily close (in C^∞) to the identity map there exists a map F such that the restriction of some iteration of F to some small ball is C^∞-conjugate to g.

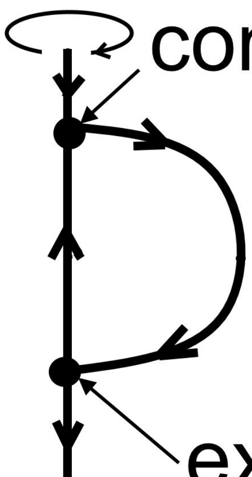
Theorem. If a smooth map F of R^n $(n \geq 3)$ has a periodic orbit with all n multipliers (the eigenvalues of the derivative of the return map) equal to 1, then by arbitrary small (in C^∞) perturbations of F one can create all n-dimensional dynamics

Proof (for $n=3$). Normal form = the time-1 map of

$$x''' + a x'' + b x' + c x = x^2 + \tau q x x' + p \tau^2 x x'' + O(\tau^3)$$



(Turaev, 2010)



contraction of area

rotation x

expansion of area

one can create

a two-dimensional periodic spot

Three-dimensional periodic spot

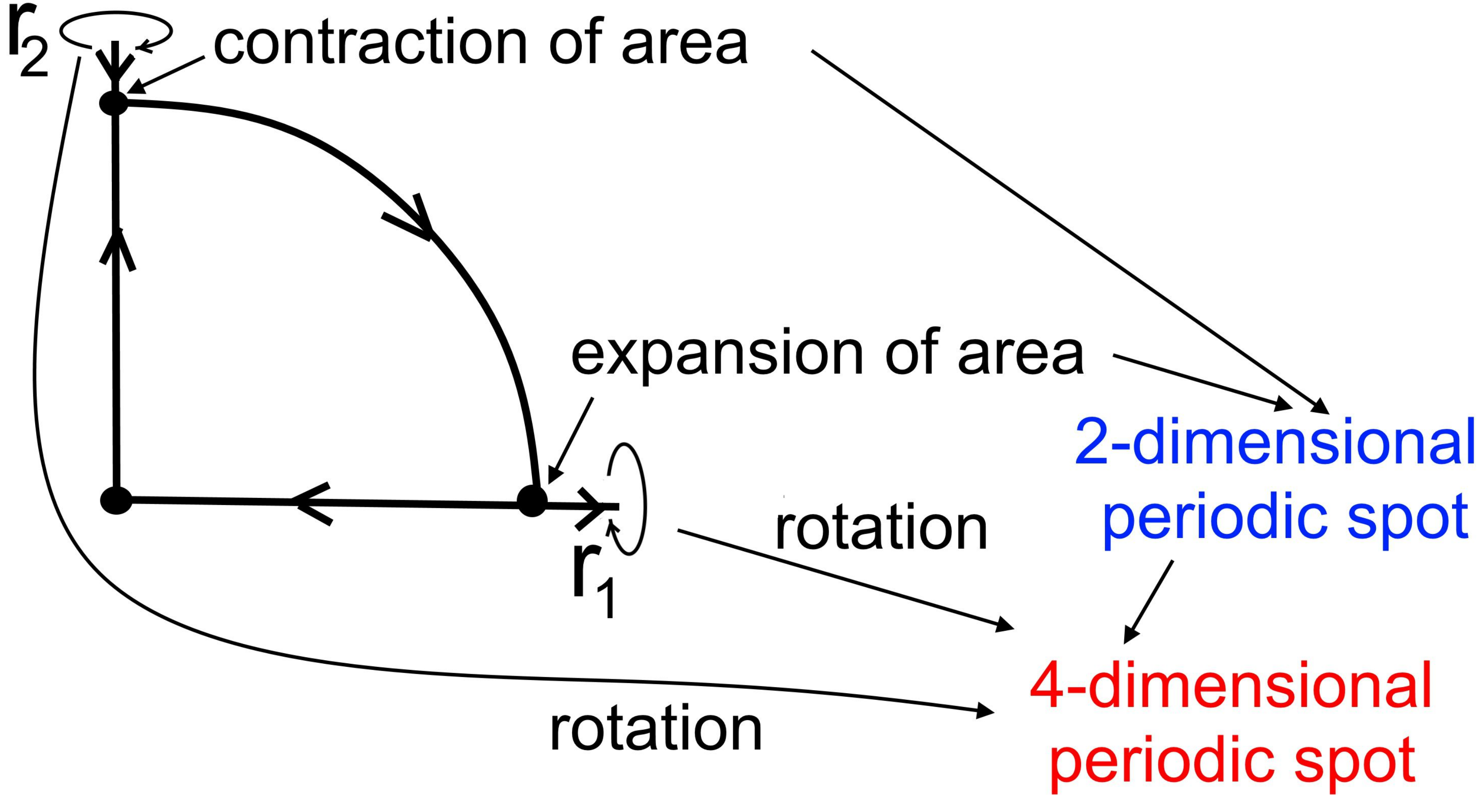
Proof for n=4

$$x'''' + a_3 x''' + a_2 x'' + a_1 x' + a_0 x = x^2 + \tau q_1 x x' + \tau^2 q_2 x x'' + \tau^3 q_3 x x''' + O(\tau^4)$$

small

$$a_2 = \omega_1^2 + \omega_2^2 + \ldots$$

$$a_0 = \omega_1^2 \omega_2^2 + \ldots$$



Proof for $n > 4$

$$x^{(n)} + (\omega_1^2 + \omega_2^2)\, x^{(n-2)} + \omega_1^2 \omega_2^2\, x^{(n-4)} = x^2 + \tau q\, x x' + O(\tau^2)$$

There exist ω_1 and ω_2 such that a small perturbation creates a two-dimensional torus with all zero characteristic exponents

By induction we obtain an (n-2)-dimensional periodic spot

times the torus

= an n-dimensional periodic spot

Conclusions:

1. Physical laws are a matter of convenience

2. Dimension is the only physical quantity

3. Would be great if someone would prove the closing lemma