Equilibrium measures for some partially hyperbolic systems

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Topological pressure

(X, d) - compact metric space,
$f : X \to X$ - homeomorphism,
$\varphi : X \to \mathbb{R}$ - continuous function (potential).

For an integer $n \geq 0$ and r>0 define:

1. $d_n(x, y) = \max\{d(f^k x, f^k y) : 0 \leq k < n\}$,
2. $B_n(x, r) = \{y : d_n(x, y) < r\}$,
3. $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k x)$,
4. $Z_n(\varphi, r) := \inf\left\{ \sum_{x\in E} e^{S_n\varphi(x)} : X \subset \bigcup_{x\in E} B_n(x, r)\right\}$.

The topological pressure of φ on X is given by

$$P(\varphi) = \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, r).$$

Equilibrium measure

Denote by $\mathcal{M}(f)$ the set of f-invariant Borel probability measures on X.

The variational principle:

$$P(\varphi) = \sup_{\mu \in \mathcal{M}(f)} \left\{ h_\mu(f) + \int \varphi \, d\mu \right\}. \tag{1}$$

We call a measure $\mu \in \mathcal{M}(f)$ an equilibrium measure for φ if it achieves the supremum in (1).

Gibbs measure

We say that a measure $\mu \in \mathcal{M}(f)$ is a Gibbs measure (or that μ has the Gibbs property) if for every $r > 0$ there is $Q = Q(r) > 0$ such that for every $x \in X$ and $n \in \mathbb{N}$, we have

$$Q^{-1} \leq \frac{\mu(B_n(x,r))}{\exp(-P(\varphi)n + S_n\varphi(x))} \leq Q. \tag{2}$$

$\varphi = 0$

Taking a constant potential function $\varphi(x) = 0$,

$$Z_n(\varphi, r) = \inf\left\{ \sum_{x\in E} e^{S_n\varphi(x)} : X \subset \bigcup_{x\in E} B_n(x,r) \right\}$$

estimates the number of Bowen balls needed to cover X. The pressure, $P(\varphi)$, corresponds to the growth rate of this number as $n \to \infty$. This is exactly the topological entropy of the map f. Therefore one has

$$P(0) = h_{top}(f)$$

and every equilibrium measure for $\varphi = 0$ is also called a measure of maximal entropy (MME).

$\varphi = -t \log J^u f$

Let now f be a $C^{1+\alpha}$ diffeomorphism which is nonuniformly (partially) hyperbolic on an invariant set Y.

$$\varphi_t(x) := -t \log J^u f(x),$$

where t is a real number and $J^u f(x) = |Df_{|E^u(x)}|$ denotes the Jacobian of f restricted to the unstable subspace $E^u(x)$ at x.

In particular, for $t = 1$ one has

$$\varphi^{\mathrm{geo}}(x) := -\log J^u f(x).$$

The Margulis–Ruelle inequality:
for any invariant Borel measure μ we have the following

$$h_\mu(f) \leq \int \log|\det Df|_{E^u(x)}|\, d\mu. \tag{3}$$

And equality holds if μ is an SRB measure (u-measure) (by results of Pesin, Ledrappier and Strelcyn, Brin)

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Partially hyperbolic set

$f : M \to M$- (local) diffeomorphism of a compact smooth manifold, $\Lambda \subset M$- compact, invariant set.

- the tangent bundle over Λ splits into two invariant and continuous subbundles $T_\Lambda M = E^{cs} \oplus E^u$;
- there is a Riemannian metric $\|\cdot\|$ on M and numbers $0 < \nu < \chi$ with $\chi > 1$ such that for every $x \in \Lambda$

$$\begin{aligned} \|Df_x v\| &\le \nu \|v\| \text{ for } v \in E^{cs}(x), \\ \|Df_x v\| &\ge \chi \|v\| \text{ for } v \in E^u(x). \end{aligned} \tag{4}$$

We require the following two conditions (among others):

(C1) f has topologically neutral center (Lyapunov stability)

Holds for example if there is a constant $L > 0$ *such that for every* $x \in \Lambda$, $v \in E^{cs}(x)$, *and* $n \in \mathbb{N}$, *we have* $\|Df_x^n v\| \le L\|v\|$;

(C2) $f|\Lambda$ is topologically transitive.

Local product structure

The distributions E^u and E^{cs}(Hertz-Hertz-Ures) can be uniquely integrated to continuous invariant foliations W^u and W^{cs} with smooth leaves.

We require the set Λ to have the following product structure:

(C3) $\Lambda = \left(\bigcup_{x\in\Lambda} V^u_{\mathrm{loc}}(x)\right) \cap \left(\bigcup_{x'\in\Lambda} V^{cs}_{\mathrm{loc}}(x')\right)$.

MME

Theorem 1 (Climenhaga, Pesin, Z.)

The following statements hold:

1. *There exists a unique MME,* μ_0. *It is ergodic and fully supported.*
2. μ_0 *has the Gibbs property: for every small* $r > 0$ *there is* $Q = Q(r) > 0$ *such that for every* $x \in$ *and* $n \in \mathbb{N}$,

$$Q^{-1} \leq \frac{\mu(B_n(x,r))}{\exp(-h_{top} n)} \leq Q. \tag{5}$$

3. μ_0 *has local product structure (locally* $\mu_0 \sim \mu_y^u \otimes \mu_y^{cs}$ *for* μ*-a.e.* $y \in \Lambda$ *).*

Counterexample - unbounded expansion in E^{cs}

let $f\colon \mathbb{T}^4 \to \mathbb{T}^4$ be given by $f(x, y) = (Ax, gy)$, where $x, y \in \mathbb{T}^2$,
$A-$ a linear toral automorphism,
g - a "slow-down" of irrational translation.

writing m for Lebesgue measure on $\mathbb{T}^2$, the measures $m \times m_\kappa$ and $m \times \delta_p$ are both measures of maximal entropy for f.

Corollary

Note that the above direct product construction of the map f together with Theorem 1 allow us to obtain a new proof of the well known result:

if $g\colon M \to M$ is a topologically transitive isometry, then g is uniquely ergodic.

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Conditions on the potential

We require that $\varphi \colon \Lambda \to \mathbb{R}$ satisfies:
u-Bowen property: there exists $Q_u > 0$ such that for every $x \in \Lambda$, $n \geq 0$, and $y \in B_n^u(x, \tau) \cap \Lambda$,

$$|S_n\varphi(x) - S_n\varphi(y)| \leq Q_u;$$

cs-Bowen property: there exist $Q_{cs} > 0$ and $r_0' > 0$ such that for every $x \in \Lambda$, $n \geq 0$, and $y \in B_\Lambda^{cs}(x, r_0')$,

$$|S_n\varphi(x) - S_n\varphi(y)| \leq Q_{cs}.$$

Let $C_{\mathrm{B}}(\Lambda)$ be the set of all functions $\varphi \colon \Lambda \to \mathbb{R}$ that satisfy both the u- and cs-Bowen properties.

Main theorem

Theorem 2 (Climenhaga, Pesin, Z.)

The following are true.

1. *There exists a unique equilibrium measure,* μ_φ *for* φ.
2. μ_φ *is ergodic and fully supported.*
3. μ_φ *has Gibbs property.*
4. μ_φ *has local product structure.*

The geometric q-potential

We consider the family of geometric q-potentials,

$$\varphi_q(x) = -q \log \det Df|E^u(x), \quad q \in \mathbb{R}.$$

We shall impose the following additional requirements:

(A1) The partially hyperbolic set $\Lambda \subset U$ is an attractor for f; that is, $\overline{f(U)} \subset U$ and $\Lambda := \bigcap_{n \geq 0} f^n(U)$.

(A2) The holonomy maps between local unstable leaves are uniformly absolutely continuous with respect to leaf volume.

Time-one map of an Anosov flow

Let $f^t : M \to M$ be an *Anosov flow*: $TM = E^s \oplus E^c \oplus E^u$,

$$E^c = \langle \dot{f} \rangle, \ \|Df^t_{|E^s}\| \leq \lambda^t, \ \|Df^{-t}_{|E^u}\| \leq \lambda^t.$$

It is known that for each $x \in M$ there is a pair of embedded discs $W^s(x)$, $W^u(x)$ such that $T_x W^s(x) = E^s(x)$ and $T_x W^u(x) = E^u(x)$. We define

$$W^{sc}(x) = \bigcup_{t \in (-r,r)} W^s(f^t(x)).$$

Having an Anosov flow $f^t : M \to M$ (with constant speed) we define a diffeomorphism $f : M \to M$ **to be the time-one map of the flow**. That is, $f(x) := f^1(x)$. If f is **topologically transitive**, the result holds for **any potential** φ **of the form** $\varphi(x) := \int_0^1 \psi(f^\tau(x))d\tau$**, where** $\psi : M \to \mathbb{R}$ **is Hölder continuous.**

Remark: time-1 map of an Anosov flow is entropy-expansive (Díaz, Fisher, Pacifico, Vieitez '12). This implies existence, but not uniqueness, of an equilibrium measure for any continuous potential φ*.*

Deducing uniqueness of MME for f^1 from uniqueness of MME for the flow f^t

If the flow is topologically mixing, we can deduce uniqueness of the MME for f from the uniqueness of the MME for the flow f^t (We would like to thank F. Rodriguez Hertz for providing us with this argument):

Assume the flow has a unique MME μ, and that (f^t, μ) is mixing. Since (f^t, μ) is mixing, then (f, μ) is ergodic.
Now if ν is any MME for $f = f^1$, then the measure $\int_0^1 f_*^t \nu \, dt$ is invariant under the flow and is the MME, since for every t the measure $f_*^t \nu$ is f-invariant and has the same (maximal) entropy as ν.
Consequently, $\int_0^1 f_*^t \nu \, dt = \mu$.
We can see that this is only possible if $f_*^t \nu = \mu$ for every $t \in [0, 1]$. Otherwise, μ could be expressed as a linear combination of f-invariant measures, which contradicts ergodicity of (f, μ). In particular, $\nu = \mu$.

Time-one map of an ergodic frame flow

M - oriented, n-dimensional of negative sectional curvature.
Define:

1. unit tangent bundle, $SM = \{(x, v) | x \in M,\ v \in T_x M,\ \|v\| = 1\}$
2. frame bundle, $FM = \{(x, v_0, v_1, \dots, v_{n-1}) | x \in M,\ v_i \in T_x M\}$ where $v_i's$ form a positively oriented orthonormal frame at x.
3. The *geodesic flow* $g^t : SM \to SM$, $g^t(x, v) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$, where $\gamma_{(x,v)}(t)$ is the unique geodesic determined by the vector (x, v).
4. The *frame flow* $F^t : FM \to FM$, $F^t(x, v_0, v_1, \dots, v_{n-1}) = (g^t(x, v_0), \Gamma^t_\gamma(v_1), \dots, \Gamma^t_\gamma(v_{n-1}))$, where Γ^t_γ is the parallel transport along the geodesic $\gamma(x, v_0)$.

F^t is a partially hyperbolic flow: $FM = E^s \oplus E^c \oplus E^u$, with $\dim E^c = SO(n-1)$. F^t acts isometrically on the center bundle.

Consider the time one map, F^1 and a potential φ of the form $\varphi(x) := \int_0^1 \psi(F^\tau(x)) d\tau$, where $\psi : M \to \mathbb{R}$ is Hölder continuous and constant along central fibers.

Remark: For an odd $n \neq 7$ existence and uniqueness of equilibrium measures for the frame flow F^t and a Hölder continuous potential which is constant on fibers was shown by Spatzier, Visscher '18.

Ergodicity of the frame flow

Proposition 1 (Brin-Gromov, Brin, Brin-Karcher, Burns-Pollicott)

Let f^t be the frame flow on an n-dimensional compact smooth Riemannian manifold with sectional curvature between $-\Lambda^2$ and $-\lambda^2$ for some $\Lambda, \lambda > 0$. Then in each of the following cases the flow and its time-1 map are ergodic:

- if the curvature is constant,
- for a set of metrics of negative curvature which is open and dense in the C^3 topology,
- if n is odd and $n \neq 7$,
- if n is even, $n \neq 8$, and $\lambda/\Lambda > 0.93$,
- if $n = 7$ or 8 and $\lambda/\Lambda > 0.99023\ldots$.

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Reference measures

Suppose

- X is a compact topological space,
- $f: X \to X$ - a continuous map and
- μ - a finite f-invariant ergodic Borel probability measure on X.

Given $Y \subset X$ with $\mu(Y) > 0$ and a measurable partition ξ of Y; let $\tilde{\mu}$ be the corresponding factor-measure on $\tilde{Y} := Y/\xi$, and $\{\mu_W^\xi : W \in \xi\}$ the conditional measures on partition elements.

Theorem 3 (Climenhaga, Pesin, Z.)

For $\tilde{\mu}$-a.e. $W \in \xi$, any probability measure ν on W such that $\nu \ll \mu_W^\xi$ has the property that $\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \nu$ converges in the weak* topology to the measure μ.

Hausdorff dimension structure

Let Z be a subset of a metric space X. We define the α-Hausdorff outer measure $m_H(Z,\alpha)$ of Z for any $\alpha \in [0,+\infty)$ as follows:

$$m_H(Z,\alpha) = \lim_{\epsilon\to 0} \inf_{\{U_i\}} \sum_i (\mathrm{diam}(U_i))^\alpha,$$

where the infimum is taken over all countable covers of Z by open sets $\{U_i\}$ such that $\mathrm{diam}(U_i) < \epsilon$.

For every $\alpha \geq 0$, $m_H(\cdot,\alpha)$ defines a Borel measure on X.

For any set $Z \subset X$ there exists a critical value α_H such that: $m_H(Z,\alpha) = \infty$ for $\alpha < \alpha_H$ and $m_H(Z,\alpha) = 0$ for $\alpha > \alpha_H$ (while $m_H(Z,\alpha_H)$ may be 0, ∞, or a finite positive number).

We call the quantity $\dim_H Z = \alpha_H$ the Hausdorff dimension of the set Z.

If $0 < m_H(X,\alpha_H) < \infty$, then its normalization defines a probability measure of maximal Hausdorff dimension on X.

Hausdorff vs Carathéodory dimension structure

Let Z be a subset of a metric space X. We define the α-Hausdorff outer measure $m_H(Z,\alpha)$ of Z for any $\alpha \in [0,+\infty)$ as follows:

$$m_H(Z,\alpha) = \lim_{\epsilon \to 0} \inf_{\{U_i\}} \sum_i (\mathrm{diam}(U_i))^\alpha,$$

where the infimum is taken over all countable covers of Z by open sets $\{U_i\}$ such that $\mathrm{diam}(U_i) < \epsilon$.

Hausdorff vs Carathéodory dimension structure

Let Z be a subset of a metric space X. We define the α-~~Hausdorff~~ Carathéodory outer measure $m_H(Z, \alpha)$ of Z for any ~~$\alpha \in [0, +\infty)$~~ $\alpha \in \mathbb{R}$ as follows:

$$m_H(Z, \alpha) = \lim_{\epsilon \to 0} \inf_{\{U_i\}} \sum_i \xi(U_i) \, (\sout{\mathrm{diam}}\ \eta(U_i))^\alpha,$$

where the infimum is taken over all countable covers of Z by ~~open sets~~ admissible sets $U_i \in \mathcal{F}$ such that ~~$\mathrm{diam}(U_i) < \epsilon$~~ $\psi(U_i) < \epsilon$.

Carathéodory Dimension Structure

Let X be a set and $\mathcal{F}$ a collection of subsets of X called admissible. Assume that there are two set functions $\eta, \psi : \mathcal{F} \to [0, \infty)$ satisfying

(H1) $\emptyset \in \mathcal{F}$; $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\eta(U), \psi(U) > 0$ for any $U \in \mathcal{F}$, $U \neq \emptyset$;

(H2) for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(U) \leq \delta$ for any $U \in \mathcal{F}$ with $\psi(U) \leq \varepsilon$;

(H3) there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, one can find a finite or countable subcollection $\mathcal{G} \subset \mathcal{F}$ covering X such that $\psi(U) \leq \varepsilon$ for any $U \in \mathcal{G}$.

Let $\xi : \mathcal{F} \to [0, \infty)$ be a set function. The collection of subsets $\mathcal{F}$ and the functions ξ, η, ψ, satisfying (H1), (H2), (H3) introduce a Carathéodory dimension structure or C-structure $\tau = (\mathcal{F}, \xi, \eta, \psi)$ on X.
η is a potential set function, ξ measures the weight, and ψ the size of $U \subset \mathcal{F}$.

For any subcollection $\mathcal{G} \subset \mathcal{F}$ let $\psi(\mathcal{G}) := \sup\{\psi(U) : U \in \mathcal{G}\}$. Given $Z \subset X$ and $\alpha \in \mathbb{R}$, define

$$m_C(Z, \alpha) := \lim_{\varepsilon \to 0} \inf_{\mathcal{G}, \psi(\mathcal{G}) \leq \varepsilon} \left\{ \sum_{U \in \mathcal{G}} \xi(U) \eta(U)^{\alpha} \right\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{F}$ covering Z.

If $m_C(\emptyset, \alpha) = 0$, the set function $m_C(\cdot, \alpha)$ becomes an outer measure on X, which induces a measure called the α-Carathéodory measure. In general, this measure may not be σ-finite or it may be a zero measure.

Furthermore, there exists $\alpha_C \in \mathbb{R}$ s.t. $m_C(Z, \alpha) = \infty$ for $\alpha < \alpha_C$ and $m_C(Z, \alpha) = 0$ for $\alpha > \alpha_C$ (while $m_C(Z, \alpha_C)$ may be 0, ∞, or a finite positive number).

The quantity $\dim_C Z = \alpha_C$ is the Carathéodory dimension of Z.

Carathéodory Structure on Local Unstable Manifolds

Fix $x_0 \in \Lambda$ and set $X := V^u(x_0) \cap \Lambda$. Fix a small number $r > 0$ and define the Bowen's u-ball by

$$B_n^u(x,r) := \{y \in V^u(x) \cap \Lambda : d(f^k(y), f^k(x)) < r \text{ for } k = 0, \dots, n\}.$$

Then define the collection $\mathcal{F}$ of admissible sets by

$$\mathcal{F} := \{\emptyset\} \cup \{B_n^u(x,r) : x \in V^u(x_0) \cap \Lambda, n \in \mathbb{N}\}.$$

Given $x \in X$ and $n > 0$, define

$$\xi(B_n^u(x,r)) := e^{S_n\varphi(x)}, \ \eta(B_n^u(x,r)) := e^{-n}, \ \psi(B_n^u(x,r)) := \frac{1}{n},$$

and also set $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\xi(\emptyset) = 1$. It is easy to see that the collection of subsets $\mathcal{F}$ and set functions ξ, η, ψ satisfy (H1), (H2), (H3), and hence, introduce a C-structure in X.

Lemma 4 (Climenhaga, Pesin, Z.)

$$\dim_C X = P(\varphi).$$

Thus we obtain the Carathéodory measure $m^u_{C,x_0}(\cdot) := m_{C,x_0}(\cdot, P)$ on X at the Carathéodory dimension $P = \dim_C X$. For every $Z \subset X$ we have

$$m^u_{C,x_0}(Z) = \lim_{N\to\infty} \inf_{\{B^u_{n_i}(x_i,r)\}} \sum_i e^{S_{n_i}\varphi(x_i)}(e^{-n})^P,$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, r)\}$ of u-Bowen's balls with $x_i \in X$, $n_i \geq N$, which cover Z that is $Z \subset \bigcup_i B^u_{n_i}(x_i, r)$.

Theorem 5 (Climenhaga, Pesin, Z.)

1. *There is $K > 0$ such that for every $x \in \Lambda$, $m_x^{\mathcal{C}}$ is a Borel measure on $V_{loc}^u(x) \cap \Lambda$ with $m_x^{\mathcal{C}}(V_{loc}^u(x) \cap \Lambda) \in [K^{-1}, K]$.*
2. *For every $x \in \Lambda$, we have $f^* m_{f(x)}^{\mathcal{C}} := m_{f(x)}^{\mathcal{C}} \circ f \ll m_x^{\mathcal{C}}$, with Radon–Nikodym derivative $e^{P(\varphi)-\varphi}$.*
3. *The family of measures $\{m_x^{\mathcal{C}}\}_{x\in\Lambda}$ has the u-Gibbs property. That is, there is $Q_0 > 0$, independent of x and n, such that for every $x \in \Lambda$ and $n \geq 0$ we have*

$$Q_0^{-1} \leq \frac{m_x^{\mathcal{C}}(B_n^u(x,r))}{e^{-nP(\varphi)+S_n\varphi(x)}} \leq Q_0. \tag{6}$$

Given a rectangle $R \subset \Lambda$ and two points $y, z \in R$, consider the holonomy map $\pi_{yz}\colon V_R^u(y) \to V_R^u(z)$ defined by $\pi_{yz}(x) := [x, z] = V_R^s(x) \cap V_R^u(z)$ for all $x \in V_R^u(y)$.

Theorem 6 (Climenhaga, Pesin, Z.)

There is a constant $C > 0$ such that for every rectangle $R \subset \Lambda$ and every $y, z \in R$, the measures $\pi_{yz}^* m_z^{\mathcal{C}} = m_z^{\mathcal{C}} \circ \pi_{yz}$ and $m_y^{\mathcal{C}}$ are equivalent on $V_R^u(y)$, with

$$C^{-1} \leq \frac{d\pi_{yz}^* m_z^{\mathcal{C}}}{dm_y^{\mathcal{C}}} \leq C$$

.

Theorem 7 (Climenhaga, Pesin, Z.)

The following are true.

1. *For every $x \in \Lambda$, the sequence of measures $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_x^{\mathcal{C}}$ is weak* convergent as $n \to \infty$ to a limiting measure μ_x, which is finite and nonzero.*
2. *The normalization of μ_x is an invariant probability measure that is independent of x. It is exactly μ_φ from Theorem 2.*
3. *For every rectangle $R \subset \Lambda$ with $\mu_\varphi(R) > 0$, the conditional measures μ_y^u generated by μ_φ on unstable sets $V_R^u(y)$ are equivalent for μ_φ-a.e. $y \in R$ to the reference measures $m_y^{\mathcal{C}}|_{V_R^u(y)}$. Moreover, there exists $C_0 > 0$, independent of R and y, such that for μ_φ-a.e. $y \in R$ we have*

$$C_0^{-1} \le \frac{d\mu_y^u}{dm_y^{\mathcal{C}}}(z) m_y^{\mathcal{C}}(R) \le C_0 \text{ for } \mu_y^u\text{-a.e. } z \in V_R^u(y). \tag{7}$$

Thank you!

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