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Estimation for trend-renewal processes

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Definition of TRP and its special cases

The trend-renewal process (TRP) is a point process,
i.e. an increasing sequence $\{T_i,\ i = 1, 2, \ldots\}$ of random variables.
It was introduced and investigated first by Lindqvist (1993), and incorporates both time trend and renewal-type behavior.

Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, such that

$$\Lambda(t) = \int_0^t \lambda(u)du < \infty$$

for each $t \geq 0$,

F be a distribution function such that $F(0) = 0$.

Definition

The process $\{T_i, i = 1, 2, \ldots\}$ is called the TRP, if the time transformed process $\{\Lambda(T_i), i = 1, 2, \ldots\}$ is the renewal process with the renewal distribution F, that is, if the random variables

$$\Lambda(T_i) - \Lambda(T_{i-1}),$$

$i = 1, 2, \ldots,$ are i.i.d. with the distribution function F.

The function F is called the renewal distribution function and $\lambda(\cdot)$ – the trend function of the TRP.

The TRP with the renewal distribution function F and the trend function $\lambda(\cdot)$ is denoted by TRP$(F, \lambda(\cdot))$.

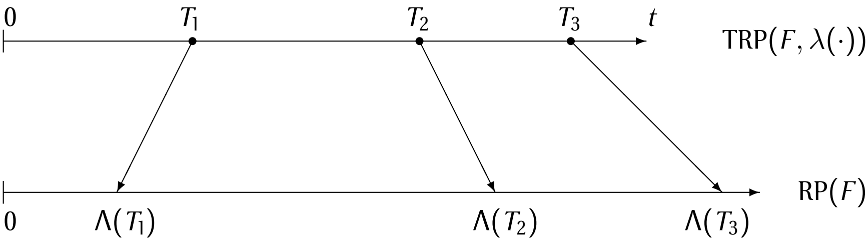

Figure 1: Illustration of the TRP

Remark

Note that the representation of the TRP$(F, \lambda(\cdot))$ is not unique. For uniqueness we usually assume a fixed scale parameter for the renewal distribution F, designed for example to make the expected value or median of F equal to one.

Remark

In the paper of Linqvist et al. (2003) the authors also assumed that

$$\lim_{t \to \infty} \Lambda(t) = \infty.$$

Table 1: Special cases of the TRPs

Model	$F(t)$	$\lambda(t)$
NHPP($\lambda(\cdot)$)	$1-\exp(-t)$	any $\lambda(t)$
RP(F)	any $F(t)$	const
NHGP($\lambda(\cdot),\kappa$)	$F(x)\sim\mathcal{G}(\kappa,1)$	any $\lambda(t)$
MGP($\varrho,\boldsymbol{\beta},\kappa$)	$F(x)\sim\mathcal{G}(\kappa,1)$	$\varrho\exp[\boldsymbol{\beta}^T\mathbf{z}(t)]$
MPLP(α,β,κ)	$F(x)\sim\mathcal{G}(\kappa,1)$	$\alpha\beta t^{\beta-1}$
WTRP($\lambda(\cdot),\kappa$)	$F(x)\sim\mathcal{W}(\kappa,1)$	any $\lambda(t)$
WPLP(α,β,κ)	$F(x)\sim\mathcal{W}(\kappa,1)$	$\alpha\beta t^{\beta-1}$

Applications of the TRPs

The class of the TRP's is a rich family of processes and were considered in the field of

- reliability (e.g., Lindqvist et al., 2003),
- finance (Zhou and Rigdon, 2008, 2011),
- medicine (Pietzner and Wienke, 2013),
- hydrology (Hurley, 1992),
- software engineering (Ishii and Dohi, 2008, Saito and Dohi, 2015),
- and to forecasts of volcanoes eruption (Bebbington, 2013).

The conditional intensity function of a point process is defined by

$$\gamma(t) = \lambda(t|\mathcal{F}_{t-}) = \lim_{\Delta t \to 0} \frac{P(\text{event in}(t, t+\Delta t)|\mathcal{F}_{t-})}{\Delta t},$$

where $\mathcal{F}_{t-} = \sigma\{N(u), u < t\}$.

For a TRP$(F, \lambda(\cdot))$ we have

$$\gamma(t) \quad = \quad z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t),$$

where $z(t)$ is the hazard rate corresponding to F.

Estimation for TRP's

Parametric estimation

- Lindqvist (1999, 2003) – ML estimation for the TRP with the power law intensity and the Weibull renewal distribution,
- Bandyopadhyay and Sen (2005) – ML estimation for the TRP with the power law intensity and the gamma renewal distribution,
- Jokiel-Rokita and Magiera (2018) – existence of the ML estimates of parameters of the TRP with log-linear intensity and gamma renewal distribution,
- Jokiel-Rokita and Magiera (2018) – estimation for the TRP with the power law intensity and the gamma renewal distribution (Z-estimators),
- Jokiel-Rokita and Skoliński (2019) – properties of the ML estimators of parameters of the TRP with log-linear intensity and gamma renewal distribution.

Semiparametric estimation

- Heggland and Lindqvist (2007) – proposed a constrained nonparametric maximum likelihood estimator of the trend function when it is monotone but unknown in the TRP with the Weibull renewal distribution.
- Lindqvist (2010) – used the kernel smoothing method to estimate the trend function when it is unknown and not necessarily monotone in the TRP with the Weibull renewal distribution.
- Jokiel-Rokita and Magiera (2012) – proposed methods of estimation of the trend functions parameters when the renewal distribution is completely unknown.
- Saito and Dohi (2016) – proposed a constrained nonparametric maximum likelihood estimator of an unknown renewal distribution with a monotone hazard rate in the TRP with the power trend function.

Nonparametric estimation

- Gàmiz and Lindqvist (2016) – consider a full nonparametric approach using kernel smoothing techniques.
  They develop an algorithm to estimate the conditional intensity function by preserving its structure in terms of the trend function and the underlying renewal process.

Estimation for the NHGP's

Next we will consider ML estimation of parameters of two special cases of the NHGP (the TRP with a gamma renewal function).
In the first case we assume that the trend function is

$$\lambda_1(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$.

The NHGP with rate function (1) was introduced by Lakey and Rigdon (1991) in [19] and it is called a modulated power law process (MPLP).

Estimation for MPLP(α, β, κ)

Let us assume that the MPLP(α, β, κ) is observed up to the nth event (failure) appears for the first time, and the values $t_1, \dots, t_n$ of the jump times $T_1, \dots, T_n$ are recorded.
Then the log-likelihood function of the MPLP(α, β, κ) is

$$\begin{aligned} \ell_{Pn}(\alpha, \beta, \kappa; \mathbf{t}) \quad = \quad & -n \log \Gamma(\kappa) - n\beta\kappa \log \alpha + n \log \beta - (t_n/\alpha)^\beta \\ & +(\beta - 1) \sum_{i=1}^{n} \log t_i + (\kappa - 1) \sum_{i=1}^{n} \log(t_i^\beta - t_{i-1}^\beta), \end{aligned}$$

where $\mathbf{t} = (t_1, \dots, t_n)$.

The likelihood equations in the model MPLP(α, β, κ) are

$$\frac{\partial \ell_{Pn}}{\partial \alpha} = (\beta/\alpha)(t_n/\alpha)^{\beta} - n\beta\kappa/\alpha = 0,$$

$$\frac{\partial \ell_{Pn}}{\partial \beta} = \frac{n}{\beta} + S_{Pn}(\mathbf{t}) + (\kappa - 1) W_{Pn}(\mathbf{t}, \beta) - n\kappa \log \alpha - \left(\frac{t_n}{\alpha}\right)^{\beta} \log\left(\frac{t_n}{\alpha}\right) = 0$$

$$\frac{\partial \ell_{Pn}}{\partial \kappa} = -n\psi(\kappa) - n\beta \log \alpha + V_{Pn}(\mathbf{t}, \beta) = 0,$$

where $\psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$ is the di-gamma function,

$$S_{Pn}(\mathbf{t}) = \sum_{i=1}^{n} \log(t_i),$$

$$W_{Pn}(\mathbf{t}, \beta) = \sum_{i=2}^{n} \frac{t_i^{\beta} \log(t_i) - t_{i-1}^{\beta} \log(t_{i-1})}{t_i^{\beta} - t_{i-1}^{\beta}} + \log(t_1),$$

$$V_{Pn}(\mathbf{t}, \beta) = \sum_{i=1}^{n} \log(t_i^{\beta} - t_{i-1}^{\beta}).$$

In the paper [7] of Black and Rigdon (1996) was noted that the numerical evaluation of the ML estimators requires the use of a complex iterative procedure based on a combined use of a first algorithm (the Nelder-Mead simplex algorithm) for approximating the location of the maximum and a second algorithm (the Newton-Raphson method) that, from a very accurate starting point, converges to the ML solution.

An alternative method of deriving the ML estimates is given in the following proposition (Jokiel-Rokita and Magiera (2018) [16]).

Proposition

The ML estimates $\hat{\alpha}_{Pn}$, $\hat{\beta}_{Pn}$, $\hat{\kappa}_{Pn}$ of the MPLP parameters α, β, κ, respectively, are

$$\hat{\alpha}_{Pn} = \frac{t_n}{(n\hat{\kappa}_{Pn})^{1/\hat{\beta}_{Pn}}}, \tag{2}$$

$$\hat{\kappa}_{Pn} = \frac{W_{Pn}(\mathbf{t}, \hat{\beta}_{Pn}) - S_{Pn}(\mathbf{t}) - n/\hat{\beta}_{Pn}}{W_{Pn}(\mathbf{t}, \hat{\beta}_{Pn}) - n\log(t_n)}, \tag{3}$$

and $\hat{\beta}_{Pn}$ is a solution to the equation

$$-n\psi(\kappa_{Pn}(\mathbf{t}, \beta)) + n\log\left[n\kappa_{Pn}(\mathbf{t}, \beta)/t_n^{\beta}\right] + V_{Pn}(\mathbf{t}, \beta) = 0. \tag{4}$$

Remark

It can be shown that a solution to equation (4) for which $\hat{\kappa}_{P_n}$ is positive, lies in the interval (β_0, ∞), where β_0 is the unique solution to the equation

$$\sum_{i=2}^{n} \frac{t_i^{\beta} \log(t_i) - t_{i-1}^{\beta} \log(t_{i-1})}{t_i^{\beta} - t_{i-1}^{\beta}} + \log(t_1) - n \log(t_n) = 0$$

with respect to β.
Moreover,

$$\beta_0 < -n / \sum_{i=1}^{n} \log(t_i / t_n) =: \beta_{start},$$

which is a good starting point for searching a solution to equation (4).

Theorem

Let $W_{1n} = n^{1/2}(\log n)^{-1}(\hat{\alpha}_{Pn} - \alpha_0)$, $W_{2n} = n^{1/2}(\hat{\beta}_{Pn} - \beta_0)$, $W_{3n} = n^{1/2}(\hat{\kappa}_{Pn} - \kappa_0)$, *where* α_0, β_0, κ_0 *are the true values of the parameters,* $\hat{\alpha}_{Pn}$, $\hat{\kappa}_{Pn}$, $\hat{\beta}_{Pn}$ *are defined by (2), (3), (4), respectively, and* $\mathbf{W}_n = (W_{1n}, W_{2n}, W_{3n})^T$.

Then, under MPLP model, $\mathbf{W}_n$ *is asymptotically (singular) normal with mean vector zero and covariance matrix*

$$\Sigma_{\mathbf{W}} = \begin{bmatrix} \frac{\alpha_0^2}{\beta_0^2\kappa_0} & \frac{\alpha_0}{\kappa_0} & 0 \\ \frac{\alpha_0}{\kappa_0} & \frac{\beta_0^2}{\kappa_0} & 0 \\ 0 & 0 & \frac{\kappa_0}{\kappa_0\psi'(\kappa_0) - 1} \end{bmatrix}.$$

Remark

Theorem 2 provides some curious insights into the behaviour of the ML estimators of the MPLP parameters.
Apart from the **singularity** *and* **nonuniform** *scalings of the ML estimators, we have also that the estimator* $\hat{\kappa}_{Pn}$ *is* **asymptotically independent** *of the estimators* $\hat{\varrho}_{Pn}$ *and* $\hat{\beta}_{Pn}$.

Estimating equations estimates of the MPLP parameters

Although Proposition 1 and Remark 1 give an efficient and reliable algorithm for finding the MLEs, in the paper Jokiel-Rokita and Magiera (2018) an alternative method of estimation of the MPLP parameters based on some properties of the MPLP is proposed. Namely, it is known from [6] that the random variables

$$U_i = \Lambda(T_i)/\Lambda(T_n),$$

where $\Lambda(t) = \alpha t^{\beta}$, are distributed according to the beta distribution $\mathcal{B}e(\kappa i, \kappa(n-i))$, and are independent of T_n. Thus

$$E(U_i) = i/n$$

and

$$\mathrm{Var}(U_i) = i(n-i)/n^2(n\kappa + 1).$$

We then propose estimating (α, β, κ) by $(\hat{\alpha}_{EE}, \hat{\beta}_{EE}, \hat{\kappa}_{EE})$, where $\hat{\beta}_{EE}$ is a solution to the equation

$$\sum_{i=1}^{n-1} \left[\left(\frac{t_i}{t_n} \right)^{\beta} - \frac{i}{n} \right] \frac{1}{i(n-i)} = 0,$$

$\hat{\kappa}_{EE}$ is a solution to the equation

$$\log(\kappa) - \psi(\kappa) = \frac{1}{n} \sum_{i=1}^{n} \log \frac{t_n^{\hat{\beta}_{EE}}}{n \left(t_i^{\hat{\beta}_{EE}} - t_{i-1}^{\hat{\beta}_{EE}} \right)},$$

and

$$\hat{\alpha}_{EE} = \frac{n \hat{\kappa}_{EE}}{t_n^{\hat{\beta}_{EE}}}.$$

In the simulation study conducted we have compared the accuracy of the estimators proposed with the maximum likelihood ones. For each chosen combination of the parameters α, β, κ, the $M = 1000$ samples of the MPLP(α, β, κ) were generated up to a fixed number $n = 30$ of jumps was reached.

- In most cases considered, the estimated mean squared error (MSE) of the estimator $\hat{\beta}_{EE}$ was smaller than the MSE of $\hat{\beta}_{ML}$.
- The MSEs of $\hat{\kappa}_{ML}$ and $\hat{\kappa}_{EE}$ were almost the same.
- On the other hand, the MSEs of $\hat{\alpha}_{EE}$ were always bigger than MSEs of $\hat{\alpha}_{ML}$.

Table 2: The RMSEs of the ML and EE estimates of α, β and κ of the MPLP(α, β, κ). The number of jumps $n = 30$.

No.	α	β	κ	se$(\widehat{\alpha}_{ML})$	se$(\widehat{\alpha}_{EE})$	se$(\widehat{\beta}_{ML})$	se$(\widehat{\beta}_{EE})$	se$(\widehat{\kappa}_{ML})$	se$(\widehat{\kappa}_{EE})$
1	1	0.5	0.5	0.9326	1.2568	0.1797	0.1496	0.1426	0.1432
2	1	1	0.5	0.9355	1.2532	0.3645	0.3135	0.1408	0.1413
3	1	2	0.5	0.8975	1.2120	0.6873	0.7218	0.1450	0.1450
4	1	0.5	1	0.8425	1.0492	0.1015	0.0887	0.3146	0.3149
5	1	1	1	0.8679	1.0817	0.2176	0.1957	0.3065	0.3069
6	1	2	1	0.8971	1.1027	0.4565	0.4578	0.3061	0.3060
7	2	0.5	2	1.4772	1.6551	0.0714	0.0669	0.7022	0.7023
8	2	1	2	1.3181	1.5013	0.1431	0.1350	0.6502	0.6504
9	2	2	2	1.2714	1.4528	0.2787	0.2901	0.6760	0.6764

Estimation for MGP(ϱ, β, κ)

In the second case we assume that the rate function of NHGP (the TRP with a gamma renewal function) is the following form

$$\lambda_2(t) = \varrho \exp(\beta t). \tag{5}$$

The NHGP with rate function (5) and a shape parameter κ will be denoted by MGP(ϱ, β, κ).

Let us notice that for the MGP(ϱ, β, κ)

$$\Lambda_2(t) = \int_0^t \lambda_2(u)du = \begin{cases} \frac{\varrho}{\beta}\Big[\exp(\beta t) - 1\Big] & \text{for} \quad \beta \neq 0, \\ \varrho t & \text{for} \quad \beta = 0, \end{cases}$$

and

$$\lim_{t\to\infty} \Lambda_2(t) = \begin{cases} -\frac{\varrho}{\beta} & \text{for} \quad \beta < 0, \\ \infty & \text{for} \quad \beta \geq 0. \end{cases}$$

Therefore, the failure truncation plan for the MGP(ϱ, β, κ) is reasonable only under the assumption $\beta \geq 0$.
For $\beta = 0$, the MGP(ϱ, β, κ) reduces to the GRP$(\kappa, 1/\varrho)$, therefore we considered ML estimation of the parameter
$\vartheta = (\varrho, \beta, \kappa) \in \Theta = (0, \infty)^3$.

The log-likelihood function of the MGP(ϱ, β, κ), observed up to the nth event appears for the first time, is the following

$$\begin{aligned}\ell_{Gn}(\varrho, \beta, \kappa; \mathbf{t}) &= n \log\left[\frac{\varrho^{\kappa}}{\Gamma(\kappa)\beta^{\kappa-1}}\right] + \beta S_{Gn}(\mathbf{t}) - \frac{\varrho}{\beta}\left[\exp(\beta t_n) - 1\right] \\ &\quad +(\kappa - 1) V_{Gn}(\beta; \mathbf{t}),\end{aligned}$$

where

$$S_{Gn}(\mathbf{t}) = \sum_{i=1}^{n} t_i,$$

$$V_{Gn}(\beta; \mathbf{t}) = \sum_{i=1}^{n} \log\left[\exp(\beta t_i) - \exp(\beta t_{i-1})\right].$$

The possible MLE's $\hat{\varrho}_{Gn}$, $\hat{\beta}_{Gn}$ and $\hat{\kappa}_{Gn}$ of MGP(ϱ, β, κ) parameters are solutions to the following system of the log-likelihood equations

$$\begin{aligned}
\frac{\partial \ell_{Gn}(\varrho, \beta, \kappa; \mathbf{t})}{\partial \varrho} &= \frac{n\kappa}{\varrho} - \frac{1}{\beta}\left[\exp(\beta t_n) - 1\right] = 0, \\
\frac{\partial \ell_{Gn}(\varrho, \beta, \kappa; \mathbf{t})}{\partial \beta} &= -\frac{n(\kappa - 1)}{\beta} + S_{Gn}(\mathbf{t}) + \frac{\varrho}{\beta^2}\left[(1 - \beta t_n)\exp(\beta t_n) - 1\right] \\
&\quad + (\kappa - 1) W_{Gn}(\beta; \mathbf{t}) = 0, \\
\frac{\partial \ell_{Gn}(\varrho, \beta, \kappa; \mathbf{t})}{\partial \kappa} &= n \log \varrho - n\psi(\kappa) - n \log \beta + V_{Gn}(\beta; \mathbf{t}) = 0,
\end{aligned}$$

where

$$W_{Gn}(\beta; \mathbf{t}) = \sum_{i=1}^{n} \frac{t_i \exp(\beta t_i) - t_{i-1}\exp(\beta t_{i-1})}{\exp(\beta t_i) - \exp(\beta t_{i-1})}.$$

Proposition

The MLE's $\widehat{\varrho}_{ML}$, $\widehat{\beta}_{ML}$ *and* $\widehat{\kappa}_{ML}$ *of the parameters* ϱ, β *and* κ *in the failure truncation procedure for the MGP*(ϱ, β, κ) *exist if and only if, given data* $\mathbf{t}$*, there exists the positive solution to the equation*

$$\begin{aligned} L_\kappa(\beta; \mathbf{t}) \quad &=: \quad \log[n\kappa(\beta; \mathbf{t})] - \psi[\kappa(\beta; \mathbf{t})] \\ &- \quad \log[\exp(\beta t_n) - 1] + \frac{1}{n} V(\beta; \mathbf{t}) = 0, \end{aligned}$$

with respect to β*, where*

$$\kappa(\beta; \mathbf{t}) = \frac{W(\beta; \mathbf{t}) - S(\mathbf{t}) - \dfrac{n}{\beta}}{W(\beta; \mathbf{t}) - n t_n \dfrac{\exp(\beta t_n)}{\exp(\beta t_n) - 1}}$$

is also positive.

Theorem

The MLE $(\widehat{\varrho}_{ML}, \widehat{\beta}_{ML}, \widehat{\kappa}_{ML})$ *of the vector parameter* (ϱ, β, κ) *for the MGP*(ϱ, β, κ) *model exists if one of the following two cases holds*

CASE 1^0*:* $D_1(\mathbf{t}) := S(\mathbf{t}) - (n+1)\frac{t_n}{2} > 0$*,*

CASE 2^0*:* $D_1(\mathbf{t}) < 0$ *and*
$\log(-t_n/2D_1(\mathbf{t})) - \Psi(-t_n/2D_1(\mathbf{t})) < -\frac{1}{n}\sum_{i=1}^{n} \log\left(n\frac{t_i - t_{i-1}}{t_n}\right)$.

Otherwise, i.e. in
CASE 3^0*:* $D_1(\mathbf{t}) < 0$ *and*
$\log(-t_n/2D_1(\mathbf{t})) - \Psi(-t_n/2D_1(\mathbf{t})) \geq -\frac{1}{n}\sum_{i=1}^{n} \log\left(n\frac{t_i - t_{i-1}}{t_n}\right)$
the MLE's do not exist.

Theorem

With probability tending to 1 as $n \to \infty$, there exists the unique sequence of roots

$$\hat{\vartheta}_n = (\hat{\varrho}_{Gn}, \hat{\beta}_{Gn}, \hat{\kappa}_{Gn}) \in \Theta = (0, \infty)^3$$

of the likelihood equations in the MGP model.

In the proof of the above theorem, we applied a version of Brouwer's fixed point theorem.

Denote by

$$\begin{aligned} Z_{1n} &= n^{\frac{1}{2}}(\log n)^{-1}(\hat{\varrho}_{Gn} - \varrho_0), \\ Z_{2n} &= n^{\frac{1}{2}}(\hat{\beta}_{Gn} - \beta_0), \\ Z_{3n} &= n^{\frac{1}{2}}(\hat{\kappa}_{Gn} - \kappa_0) \end{aligned}$$

the centred and scaled ML estimators of the MGP parameters, where ϱ_0, β_0, κ_0 are the true values of the MGP parameters, and $\mathbf{Z}_n = (Z_{1n}, Z_{2n}, Z_{3n})'$.

Theorem

The vector $\mathbf{Z}_n$ *is asymptotically (singular) normal with mean vector zero and covariance matrix*

$$\Sigma_{\mathbf{Z}} = \begin{bmatrix} \frac{\varrho_0^2}{\kappa_0} & \frac{\varrho_0\beta_0}{\kappa_0} & 0 \\ \frac{\varrho_0\beta_0}{\kappa_0} & \frac{\beta_0^2}{\kappa_0} & 0 \\ 0 & 0 & \frac{\kappa_0}{\kappa_0\psi'(\kappa_0)-1} \end{bmatrix}.$$

Remark

As in the case of the MPLP, we have that the ML estimators in the MGP model are asymptotically normal, but with **different convergence rates***, and with* **a singular covariance matrix***. Moreover, the estimator* $\hat{\kappa}_{Gn}$ *is* **asymptotically independent** *of the estimators* $\hat{\varrho}_{Gn}$ *and* $\hat{\beta}_{Gn}$.

A sketch of the proof of Theorem 5

- In the first step, we show that the suitable scaled matrix of the second derivatives of the log-likelihood function tends in probability, when $n \to \infty$, to a singular matrix. Therefore, we can't use standard methods to prove asymptotic normality of the ML estimators in the model considered.
- We prove some convergence results involving T_i's.
- We partition the parameter $\vartheta = (\varrho, \beta, \kappa)$ as (ϱ, θ) and reduce the problem to two-dimensional.
- We also apply distributional properties of the MGP and the delta method.

Conclusions and some prospect

- In the presentation we have considered parameter estimation in two special cases of inhomogeneous gamma process model for analysis of a single repairable system.
- We have presented cases of asymptotics that are quite nonstandard and are in stark contrast with what one typically encounters in ML estimation theory.
- We are also interested in the ML estimation of WTRP parameters. Is the asymptotic covariance matrix of the ML estimator nonsingular also in this case?

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