Quantum posets and Tevelev degrees

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Joint with P.-E. Chaput, L. Mihalcea, N. Perrin (posets)

R. Pandharipande (Tevelev degrees)

Flag varieties

$X = G/P$ Flag variety.

$B \subseteq P \subseteq G$. $\quad W_P \leq W$ Weyl groups.

$W^P \subseteq W$ min. representatives of cosets in W/W_P.

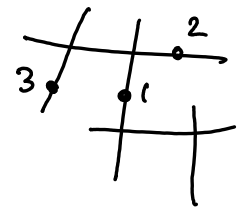
Schubert varieties

$u \in W$: $\quad X_u = \overline{Bu.P}\,, \quad X^u = \overline{B^- u.P} \quad \subseteq X$

$u \in W^P \Rightarrow \dim(X_u) = \operatorname{codim}(X^u, X) = \ell(u)$.

Curve neighborhoods

$M_d = \overline{\mathcal{M}}_{0,3}(X,d)$ Kontsevich moduli space. $\qquad d \in H_2(X,\mathbb{Z})$.

$= \{ \text{stable } f : C \longrightarrow X \text{ of degree } d \}$ $\qquad C =$ 

$M_d(X_u) = ev_1^{-1}(X_u) \subseteq M_d$

$\Gamma_d(X_u) = ev_3(M_d(X_u)) = \overline{\text{union}}$ of (rational) curves $C \subseteq X$ of deg. d meeting X_u.

$M_d(X_u, X^v) = ev_1^{-1}(X_u) \cap ev_2^{-1}(X^v)$

$\Gamma_d(X_u, X^v) = ev_3(M_d(X_u, X^v)) = \overline{\text{union}}$ of $C \subseteq X$ of deg. d connecting X_u, X^v.

<u>Quantum cohomology</u>

$QH(X) = H^*(X,\mathbb{Z}) \otimes \mathbb{Z}[q]$ algebra over $\mathbb{Z}[q]$

$$[X_u] * [X^v] = \sum_{d\geq 0} (X_u * X^v)_d \, q^d$$

where $(X_u * X^v)_d = (ev_3)_* [M_d(X_u, X^v)]$

<u>Note</u>: $\operatorname{coeff}\big((X_u * X^v)_d \,;[X^w]\big) = \langle X_u, X^v, X_w \rangle_d$

$= \# \; f : \mathbb{P}^1 \longrightarrow X$ of deg. d s.t.

$0, 1, \infty \mapsto X_u, X^v, g.X_w$ $\quad$ ($g \in G$ general)

<u>Note</u>: $[X^u] = [X_{u^\vee}]$, $\quad u^\vee = w_0 u w_{0,P} \in W^P$

Poincaré dual element.

Question: Which powers $q^d \in [X^u] * [X^v]$?

Fulton – Woodward , Postnikov :

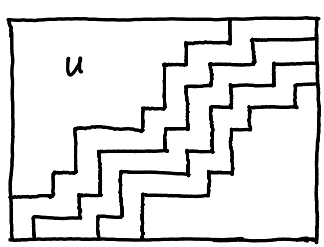
- $\exists$ unique minimal power $q^{d_{min}(u,v)}$.

$d_{min}(u,v)$ = min. degree of curve connecting $X_{u^\vee}$, X^v.

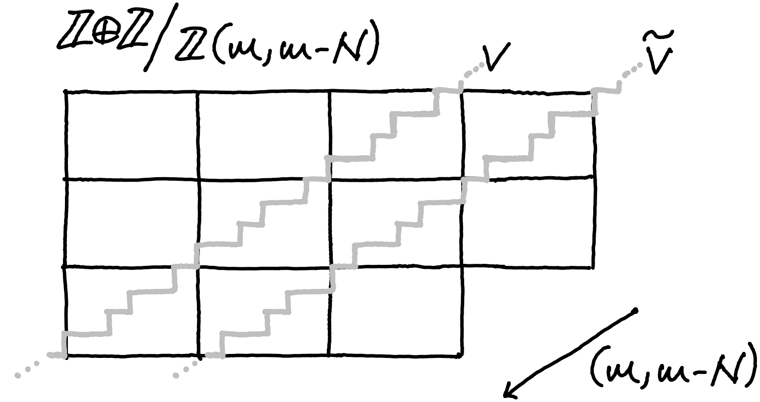
$X = Gr(m,N)$ Grassmannian of type A :

- q-degrees in $[X^u] * [X^v]$ form an interval.
- $q^d \in [X_u] * [X^v] \iff v \leq u(d) \leq \tilde{v}$ in Postnikovs cylinder:

$u(d) = u + d$ rim-hooks :



$d=3$



Example $X = Fl(\mathbb{C}^6)$. $w = 164532$.

The set $\{ d \in H_2(X) \mid q^d \in [X^w] * [X^w] \}$ is not saturated and has no unique max element.

Cominuscule flag varieties

$Gr(m, N)$, $LG(N, 2N)$, $OG(N, 2N)$, Q^N, E_6/P_6, E_7/P_7.

Main result: (BCMP)

Assume X is cominuscule. Then

- q-degrees in $[X_u] * [X^v]$ form an interval.
- Determined by type-independent generalization of Postnikov's cylinder.

Quantum equals classical

(B, Kresch, Tamvakis, Chaput, Manivel, Perrin, Mihalcea)

$X = Gr(m, N)$. $C \subseteq X$ curve.

Def $\operatorname{Ker}(C) = \bigcap_{V \in C} V$, $\operatorname{Span}(C) = \sum_{V \in C} V \subseteq \mathbb{C}^N$.

Exercise $C \subseteq X$ general of deg. d $\Rightarrow$

$$(\operatorname{Ker}(C), \operatorname{Span}(C)) \in Y_d = Fl(m-d, m+d; N)$$

$$\begin{array}{rcl} Z_d & = & Fl(m-d, m, m+d; N) \xrightarrow[p]{} X \\ & & \downarrow q \\ Y_d & = & Fl(m-d, m+d; N) \end{array}$$

Note: $q(p^{-1}(X_u)) = \left\{ \begin{array}{l} \text{ker-span pairs of general } C \subseteq X \\ \text{of degree } d \text{ meeting } X_u. \end{array} \right\}$

Thm $\langle X^u, X^v, X^w \rangle_d = \int_{Y_d} q_* p^* [X^u] \cdot q_* p^* [X^v] \cdot q_* p^* [X^w]$.

Restatement

$$\boxed{Z_d = Fl(m-d, m, m+d; N)}$$

Def $Z_d(X_u, X^v) = q^{-1}q\,p^{-1}(X_u) \cap q^{-1}q\,p^{-1}(X^v)$.

Thm $\exists$ birational map (for range of degrees d)

$$\varphi : M_d(X_u, X^v) \overset{\approx}{\dashrightarrow} Z_d(X_u, X^v)$$

$$\varphi(f) = (\mathrm{Ker}(f),\ ev_3(f),\ \mathrm{Span}(f))$$

Cor $\Gamma_d(X_u, X^v) = ev_3(M_d(X_u, X^v)) = p(Z_d(X_u, X^v))$ projected Richardson variety

Cor $(X_u * X^v)_d = p_*[Z_d(X_u, X^v)]$

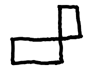
$$= \begin{cases} [\Gamma_d(X_u, X^v)] & \text{if } p : Z_d(X_u, X^v) \longrightarrow \Gamma_d(X_u, X^v) \text{ birational} \\ 0 & \text{otherwise.} \end{cases}$$

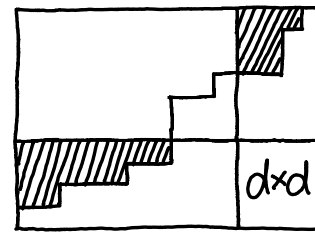
Fibers of $p : Z_d \longrightarrow X_d$:

$\boxed{Z_d = Fl(m-d, m, m+d; N)}$

$F_d = Gr(m-d, m) \times Gr(d, N-m)$

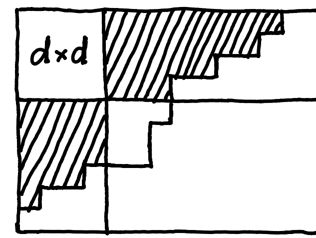
General fibers of $p : Z_d(X_u) \longrightarrow \Gamma_d(X_u)$:

$(F_d)_{u_d}$ where $u_d = u \cap$ 

u 

General fibers of $p : Z_d(X^v) \longrightarrow \Gamma_d(X^v)$:

$(F_d)^{v^d}$ where $v^d = v \cap$ 

V 

General fibers of $p : Z_d(X_u \cap X^v) \longrightarrow \Gamma_d(X_u \cap X^v)$:

<u>THM:</u> <u>Semi-transversal intersection</u> of $(F_d)_{u_d}$ and $(F_d)^{v^d}$ in F_d.

<u>THM:</u> $= (F_d)_{u_d} \cap (F_d)^{u_d \cap v^d}$

Interval of q-degrees

<u>Cor</u> X cominuscule flag variety, $u, v \in W^P$.

Assume $\Gamma_d(X_u, X^v) \neq \emptyset$ and $\boxed{d \times d} \subseteq u^\vee, v$. Then

$$q^d \in [X_u] * [X^v]$$
$$\Updownarrow$$
$$p : Z_d(X_u, X^v) \longrightarrow \Gamma_d(X_u, X^v) \text{ birational}$$
$$\Updownarrow$$
$$(F_d)_{u_d} \cap (F_d)^{u_d \cap v^d} = \text{point}$$
$$\Updownarrow$$
$$u_d \leq v^d.$$

<u>Cor</u> The q-degrees in $[X_u] * [X^v]$ form an interval.

<u>Proof</u>: Check that $u_{d+1} \leq v^{d+1} \Rightarrow u_d \leq v^d$.

Minimal and maximal powers of q

$\boxed{\text{Consider } [X_u] * [X^v]}$

Recall: $d \geq d_{min}(u^\vee, v)$

$\Updownarrow$ $\exists$ curve $C \subseteq X$ of degree d connecting X_u, X^v.

$\Updownarrow$ $\Gamma_d(X_u) \cap X^v \neq \emptyset$.

Thm $\Gamma_d(X_u) = X_{u(d)}$ Schubert variety, $u(d) \in W^P$.

X cominuscule $\Rightarrow$ $u(d) = u + d$ cominuscule rim-hooks.

Cor $d_{min}(u^\vee, v) = \min\{d : u(d) \geq v\}$

Thm (Chaput-Manivel-Perrin)

$[pt] * [X^v] = q^{d_{min}(v, pt)} [X^{w_0^P v}]$; $w_0^P \in W^P$ max. elt.

$d_{max}(u^\vee, v) = d_{min}(v, pt) - d_{min}((w_0^P v)^\vee, u)$

Partial order on basis

$\mathbb{Z}$-basis of $QH(X)_q$: $\mathcal{B} = \{ q^d [X^u] \mid u \in W^P,\ d \in \mathbb{Z} \}$

<u>Def</u> $q^e [X^v] \leq q^d [X^u] \iff$ X_u and X^v are connected by rational curve of deg. $d-e \geq 0$.

<u>Theorem</u> X cominuscule.

$$q^d \in [X_u] * [X^v] \iff [X^v] \leq q^d [X^u] \leq [pt] * [X^v].$$

Diagrams of boxes

$X = G/P$ cominuscule.

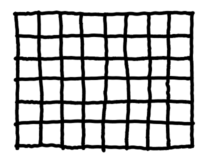
<u>Thm</u> (Proctor) $(W^P, \leq)$ is a distributive lattice.

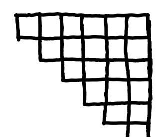
$u \cup v$ defined by $X^{u \cup v} = X^u \cap X^v$.

$u \cap v = (u^\vee \cup v^\vee)^\vee$.

<u>Def</u> $\mathcal{P}_X \subseteq W^P$ subset of join irreducible elements.

$W^P \longleftrightarrow$ (lower) order ideals in $\mathcal{P}_X$.

<u>Examples:</u> $X = Gr(m, N)$. $\mathcal{P}_X =$  $m \times (N-m)$

$X = LG(N, 2N)$ $\mathcal{P}_X =$  N rows

Quantum poset

$\mathcal{B} = \{ q^d[X^u] \}$ $\qquad$ $q^e[X^v] \le q^d[X^u] \iff \Gamma_{d-e}(X_u, X^v) \neq \emptyset$

Thm $(\mathcal{B}, \le)$ is a distributive lattice. For $e \le d$:

$$q^e[X^v] \cap q^d[X^u] = q^e[X^{v \cap u(d-e)}]$$

$$q^e[X^v] \cup q^d[X^u] = q^d[X^{v(e-d) \cup u}]$$

Def $\hat{P}_X \subseteq \mathcal{B}$ subset of join-irreducible elements.

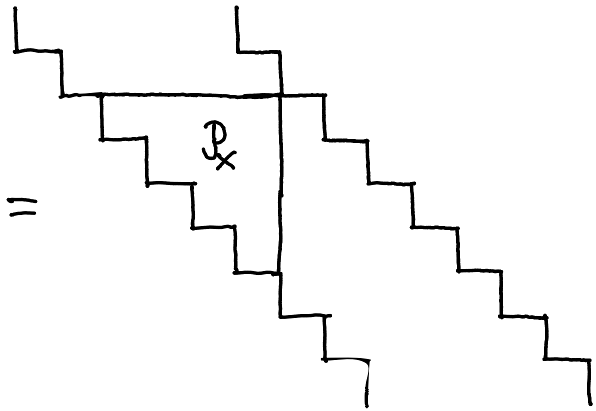
$\mathcal{B} \longleftrightarrow$ (non-empty, proper, lower) order ideals in $\hat{P}_X$.

Also true: $P_X \subseteq \hat{P}_X$.

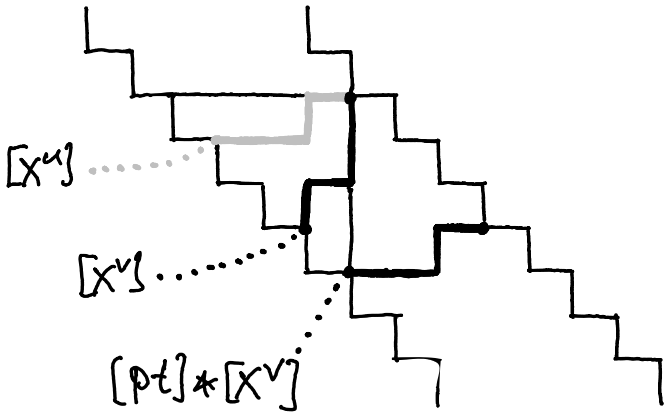
Example $X = Gr(m, N)$. $\hat{P}_X = \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z}(m, m-N)$.

Example $\quad X = LG(4,8)$.

$\hat{\mathcal{P}}_X =$



Example $\quad X = LG(4,8)$.



Multiply by q:

translate one box SE.

Multiply by $[pt]$:

reflect border in diagonal and attach to right side of P_X.

$u = (3)$, $\quad v = (4,3,1)$

$$[X_u] * [X^v] = q^2 [X^{(3,2)}] + q^2 [X^{(4,1)}] + q^3$$

Tevelev Degrees

Joint with Pandharipande.

X smooth projective variety $/\mathbb{C}$. $\quad d \in H_2(X;\mathbb{Z})$.

$\phi : \overline{M}_{g,n}(X,d) \longrightarrow \overline{M}_{g,n} \times X^n$

$\phi_* [\overline{M}_{g,n}(X,d)]^{vir} \in H^*(\overline{M}_{g,n} \times X^n)$

"cohomological field theory of X."

$$\operatorname{virdim} \overline{M}_{g,n}(X,d) = \int_d c_1(T_X) + (3 - \dim X)(g-1) + n$$

Assume $(*)$ $\operatorname{virdim} \overline{M}_{g,n}(X,d) = \dim(\overline{M}_{g,n} \times X^n)$

Then $\operatorname{virdeg}(\phi)$ is a Tevelev degree.

Note: $(*) \Leftrightarrow \deg(q^d P) = \deg(P^{n+g})$ in $QH(X)$, $\quad P = [pt]$.

Deformed Euler characteristics

$\{A_1, A_2, \dots, A_K\}$ basis of $H^*(X) = H^*(X, \mathbb{Q})$.

$\{B_1, B_2, \dots, B_K\}$ dual basis: $A_i \smile B_i = P \in H^*(X)$.

<u>Def</u> $\Delta = \sum_{i=1}^{K} B_i * A_i \in QH(X)$.

<u>Note:</u> $\Delta = \chi(X)\, P + q\text{-terms}$. $\chi(X)$ = top. Euler char.

<u>Thm</u> (Pandharipande)

$\circledast \Rightarrow \operatorname{virdeg}(\phi) = \operatorname{coeff}(P^n \Delta^g ; q^d P)$

Work in progress: Compute RHS!

Example:

$X = \mathbb{P}^N$. Basis $\{1, H, H^2, \dots, H^N\}$, H hyperplane class.

$$\Delta = \sum_{i=0}^{N} H^i * H^{N-i} = (N+1)\,\mathrm{p} \quad \text{since } \deg(q) = \deg(H^{N+1}).$$

$$\circledast \Leftrightarrow \deg(\mathrm{p}^n \Delta^g) = \deg(q^d \mathrm{p}) \Leftrightarrow (N+1)d = N(n+g-1)$$

$$\Rightarrow \operatorname{coeff}(\mathrm{p}^n \Delta^g ; q^d \mathrm{p}) = (N+1)^g$$

Tevelev degrees of cominuscule varieties

$X = G/P$ cominuscule.

$$\Delta = \sum_{u \in W^P} [X_u] * [X^u] \in QH(X).$$

Note: $\mathcal{P} = [pt]$ is invertible in $QH(X)_q$.

$\mathcal{P}$ has finite order in $QH(X)/\langle q=1 \rangle$.

Cor $\operatorname{coeff}(\mathcal{P}^u \Delta^g ; q^d \mathcal{P}) = \operatorname{coeff}(\Delta^g ; q^d \mathcal{P}^{1-u})$

depends on u modulo $\operatorname{ord}(\mathcal{P})$.

Example $(g=0)$ ⊛ $\deg(\mathcal{P}^u) = \deg(q^d \mathcal{P}) \Rightarrow$

$$\operatorname{coeff}(\mathcal{P}^u ; q^d \mathcal{P}) = \operatorname{coeff}(\mathcal{P}^{u-1} ; q^d) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod{\operatorname{ord}(\mathcal{P})} \\ 0 & \text{else.} \end{cases}$$

Genus one

Note: $\quad \mathrm{coeff}([X_u] * [X^v] \,;\, P) = \delta_{u,v}$

$$\Rightarrow \quad \mathrm{coeff}([X_u] * [X^u] * P^u \,;\, q^d P)$$
$$= \mathrm{coeff}([X^u] * P^u \,;\, q^d [X^u]) \in \{0,1\}.$$

Thm $(g=1)$ $\quad$ $\circledast$ $\deg(P^u \Delta) = \deg(q^d P) \Rightarrow$

$$\mathrm{coeff}(P^u \Delta \,;\, q^d P) = \#\{u \in W^P \mid P^u * [X^u] = q^d [X^u]\}$$

Example $X = LG(N, 2N)$, $g = 1.$

$p^2 = q^N$ in $QH(X)$.

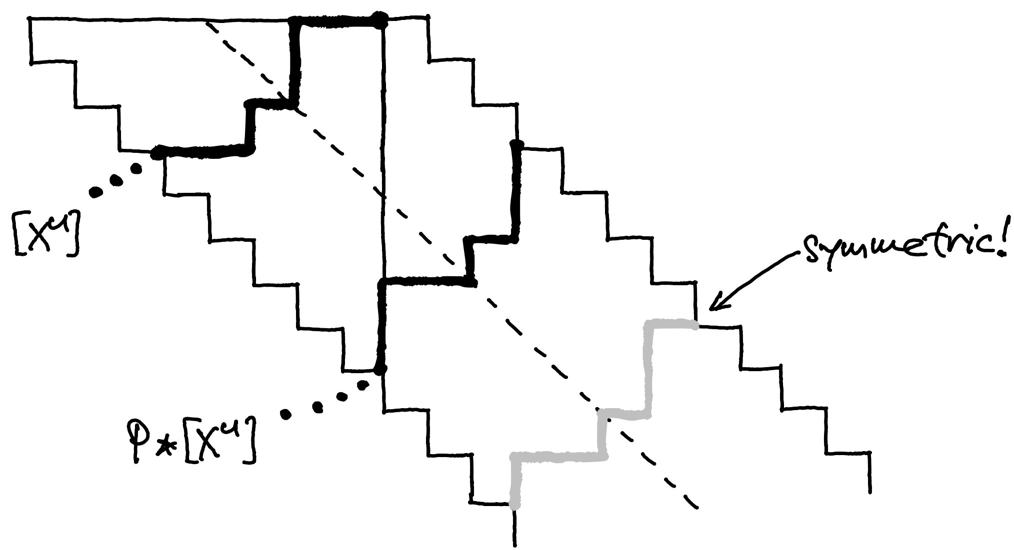
$$\mathrm{coeff}(p^u \Delta; q^d p) = \mathrm{coeff}(\Delta; q^d p^{1-u})$$

WLOG $u = 0$ or $u = 1$.

$u=0$: $\mathrm{coeff}(\Delta; p) = \chi(X) = \# W^P = 2^N$.

$u=1$: (N even)

$$\mathrm{coeff}(p\Delta; q^{N/2} p) = \# \{ u \in W^P \mid p * [X^u] = q^{N/2} [X^u] \}$$



$$P * [X^u] = q^{N/2} [X^u]$$

$\Updownarrow$

u symmetric about diagonal.

$$\mathrm{coeff}(\Delta P \,;\, q^{N/2} P) = \#\{u \text{ with symmetric border}\} = 2^{N/2}.$$

Quadric hypersurface

$X \subseteq \mathbb{P}^{N+1}$ quadric hypersurface of dim. $N \geq 3$.

$\deg(q) = \deg(P)$ in $QH(X)$.

$$\Delta = (N+a)P + (N-a)q$$

where $a = \begin{cases} 1 & \text{if } N \text{ odd} \\ 2 & \text{if } N \text{ even.} \end{cases}$

<u>Thm</u> $\circledast$ $d+1 = n+g \Rightarrow$

$$\operatorname{coeff}(P^n \Delta^g ; q^d P) = \frac{(2N)^g - (-1)^{n+g}(2a)^g}{2}$$

Hypersurface

$X \subseteq \mathbb{P}^{N+1}$ hypersurface of degree $m \geq 3$, dimension $N \geq 3$.

$H_2(X;\mathbb{Z}) = \mathbb{Z} \Rightarrow$ one deformation parameter q.

$$\frac{1}{2}\deg(q) = \int_{\text{line}} c_1(T_X) = N+2-m.$$

<u>Thm</u> Assume $\deg(P^n\Delta^g) = \deg(q^d P)$ and $\deg(q^2) > \deg(P)$.

$$\operatorname{coeff}(P^n\Delta^g\,;\,q^dP) = ((m-1)!)^n\,((N+2-m)\,m^{m-1})^g\,m^{(d-n-g)m-1}$$

Complete intersection

$X = V(f_1, \dots, f_L) \subseteq \mathbb{P}^{N+L}$ complete intersection of dim. $N \geq 3$ and degrees $(m_1, \dots, m_L)$.

$$|m| = \sum_{i=1}^{L} m_i \ ; \qquad m^{am+b} = \prod_{i=1}^{L} m_i^{am_i+b} \ ; \qquad (m-1)! = \prod_{i=1}^{L} (m_i-1)!$$

$$\tfrac{1}{2} \deg(q) = N+L+1-|m|$$

Thm Assume $\deg(P^n \Delta^g) = \deg(q^d P)$,
$\deg(q^2) > \deg(P)$,
$|m| \geq L+2$ (X not quadric).

$$\mathrm{coeff}(P^n \Delta^g ; q^d P) = ((m-1)!)^n \left((N+L+1-|m|)\, m^{m-1}\right)^g m^{(d-n-g)m+1}$$