FINITE FREE RESOLUTIONS, ROOT SYSTEMS AND SCHUBERT VARIETIES

ongoing projects with L. Guerrieri, J. Torres, J. Weyman and X. Ni } @ UJ

IMPANGA 20 Bedlewo

FINITE FREE RESOLUTIONS

R Noetherian ring, eg $R = k[x_1 \_\_ x_n]$

A finite free resolution

$$\mathbb{F}_\bullet : \quad 0 \to F_m \xrightarrow{d_m} F_{m-1} \xrightarrow{d_{m-1}} \cdots \to F_2 \to F_1 \xrightarrow{d_1} F_0$$

$F_i = R^{f_i}$ $\qquad$ d_i $\quad$ $f_{i-1} \times f_i$ matrix with entries in R

1) $\mathbb{F}_\bullet$ is a complex, i.e. $d_i \circ d_{i+1} = 0$,

2) $\mathbb{F}_\bullet$ is acyclic, i.e. $\ker d_i = \operatorname{im} d_{i+1}$.

Note: d_1 is not an epi

$$\to F_1 \to F_0 \to \underset{\substack{\parallel \\ F_0/\operatorname{im}(d_1)}}{M} \to 0 \qquad R\text{- module}$$

$\mathbb{F}_\bullet$ is "a resolution of M".

Graded case

Hilbert: Every graded R-module has a finite free resolution.

$$R = k[x_1 \ldots x_n] = \bigoplus_{i \geq 0} R_i \qquad \text{a graded ring}$$

R_i = span of monomials of degree i

$R_i \cdot R_j \subseteq R_{i+j}$

$$M = \bigoplus_{i \geq 0} M_i \qquad M_i \ \ k\text{-vector spaces}$$

M is a graded R-module if $R_i \cdot M_j \subset M_{i+j}$.

Ex: $R = k[x_1, x_2]$ $\qquad M = k = R/(x_1, x_2) = k \oplus 0 \oplus \cdots$

$$0 \to R \xrightarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} R^2 \xrightarrow{(x_1, x_2)} R \to M \to 0$$

$$1 \mapsto \bar{1}$$

→ smallest example of Koszul complex.

Local case

$(R, \mathfrak{m})$ local $:\Leftrightarrow$ $\exists$ a unique maximal ideal $\mathfrak{m}$

regular $:\Leftrightarrow$ $\dim R = d$ and $\mathfrak{m} = (x_1, \ldots, x_d)$

Thm (Auslander - Buchsbaum - Serre)

$(R, \mathfrak{m})$ regular $\Leftrightarrow$ every finitely generated R-module M has a finite free resolution

$\Leftrightarrow$ R-module $R/\mathfrak{m}$ has a finite free resolution.

Non-ex: $R = k[x]/(x^3)$ $M = R/(x) = k$

$$\cdots \to R \xrightarrow{x^2} R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \to M \to 0$$
$$1 \mapsto \bar{1}$$

periodic

($\dim R = 0$ and $\mathfrak{m}$ is generated by one element.)

Finite free resolutions of length 2

Hilbert: $R = k[x_1 \text{---} x_n] \supset I$ ideal

Suppose there exists an exact complex

$$0 \to F_2 \xrightarrow{d_2} \underset{\overset{\|}{R^m}}{F_1} \xrightarrow{d_1} R \to R/I \to 0 \qquad d_1 = (y_1 - y_m)$$

Then $F_2 = R^{m+1}$ and there exists a non zero-divisor A st $y_i = (-1)^i \cdot A \cdot \Delta_i$

$$d_2 = \begin{pmatrix} x_{11} & \text{---} & x_{1,m} \\ | & & \\ x_{m+1,1} & & x_{m+1,m} \end{pmatrix},$$

Δ_i = determinant of the submatrix of d_2 by deleting i-th row.

Burch: similar for local rings.

$\to$ gives a model for all FFR of length 2.

Acyclicity Criteria

When is a complex

$\mathbb{F}_\bullet : \quad 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$ exact ?

Define rank of d_i $\quad f_{i-1} \times f_i$ matrix

$$rk(d_i) := \max \{ r_i \mid r_i \times r_i \text{ minor of } d_i \text{ is not zero} \}$$

Moreover, $d : F \to G$

$$I_s(d) := \text{ideal in } R \text{ generated by } s \times s \text{ minors of } d$$

Buchsbaum - Eisenbud :

$F_\bullet$ is a FFR $\iff$ 1) $r_i + r_{i-1} = rk F_i = f_i \ \forall i$

(in particular $f_n = r_n$)

2) depth $I_{r_i}(d_i) \geq i$ –

Perfect ideals

$\mathrm{depth}(I) :=$ length of a (hence any) maximal regular sequence in I.

$\mathrm{pd}_R(R/I) :=$ length of a (hence any) free minimal free resolution of R/I.

In general, $\mathrm{depth}(I) \leq \mathrm{pd}_R(R/I)$

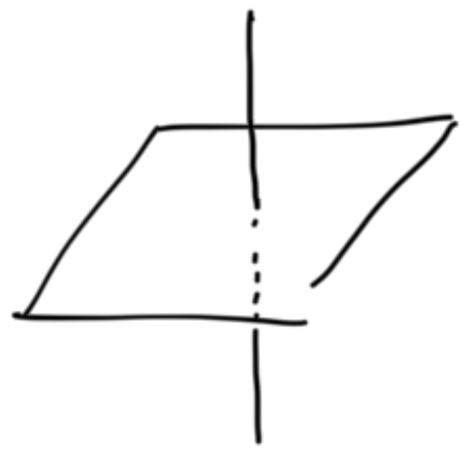
I is perfect $:\Longleftrightarrow \mathrm{depth}(I) = \mathrm{pd}_R(R/I)$

Ex: I is a complete intersection, $R = k[x_1 - x_n]$
$:\Longleftrightarrow I = (f_1 - f_r)$ a regular sequence

$$\mathrm{depth}(I) = r = \mathrm{pd}_R(R/I)$$

Koszul complex on $(f_1, -, f_r)$

Non-ex: $R = k[x,y,z] \supset I = (xy, xz)$



$I \subset (x)$ depth $I = 1$

$\mathrm{pd}(R/I) = 2$, since

$$0 \to R \xrightarrow[\binom{z}{-y}]{} R^2 \xrightarrow[(xy,\, xz)]{} R \to R/I \to 0$$

Ex: Hilbert - Burch

$$0 \to R^{n-1} \xrightarrow{d_2} R^n \xrightarrow{d_1} R \quad (\to R/I \longrightarrow 0)$$

$$d_1 = I_{n-1}(d_2)$$

I is perfect of codimension 2.

Q: perfect of codim 3 ?

Gorenstein ideals

I Gorenstein $\iff$ I is perfect and $\mathbb{F}_\bullet$ is self-dual.

Buchsbaum–Eisenbud: R polynomial ring or regular local ring

If $I \subset R$ is a Gorenstein ideal of codim 3,

$$0 \to R \xrightarrow{d_3} R^n \xrightarrow{d_2} R^n \longrightarrow R$$

then $n = 2m+1$ and

d_2 skew-symmetric $(2m+1) \times (2m+1)$ - matrix

$d_1 = \left(Pf_{\hat{1}}(d_2), \_\_\_, Pf_{\widehat{2m+1}}(d_2) \right)$ submaximal Pfaffians of X.

Q: Gorenstein ideals of codim 4?

Generic ring

R Noetherian

$$\mathbb{F}_\bullet : \quad 0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \quad \to M \to 0$$

Format of $\mathbb{F}_\bullet$ $\quad (f_0, f_1, f_2, f_3)$, $f_i = \operatorname{rk} F_i$

$r_i = \operatorname{rk} d_i$

$$f_3 = r_3 \ , \quad f_2 = r_3 + r_2 \ , \quad f_1 = r_2 + r_1 \ , \quad f_0 = r_1$$

A pair $(R_{gen}, \mathbb{F}_{gen})$ is a generic pair if :

1) $\mathbb{F}_{gen}$ is acyclic,

2) $\forall\ (S, \mathbb{F})$ $\quad \mathbb{F}$ acyclic complex of format (f_0, f_1, f_2, f_3)

$\exists \quad \varphi : R_{gen} \to R \quad$ s.t. $\quad \mathbb{F} = \mathbb{F}_{gen} \otimes_{R_{gen}} R.$

Existence of R_{gen} ? When is R_{gen} Noetherian ?

Hochster: yes for all resolutions of length 2

Given a format $0 \to R^{f_2} \to R^{f_1} \to R^{f_0}$

there exists a Noetherian ring R_{gen} with

$$\mathbb{F}_{gen} : \quad 0 \to R_{gen}^{f_2} \xrightarrow{d_2} R_{gen}^{f_1} \xrightarrow{d_1} R_{gen}^{f_0}$$

st 1) $\mathbb{F}_{gen}$ is acyclic

2) every acyclic complex of the same format comes from $\varphi : R_{gen} \to R$.
The homomorphism is unique.

The construction of R_{gen} is explicit:
one starts with a "generic complex" and adds fractions (Buchsbaum - Eisenbud multipliers).

Generic ring for $n=3$

Pragacz – Weyman: similar procedure by adding BE multipliers and factoring through their relations.

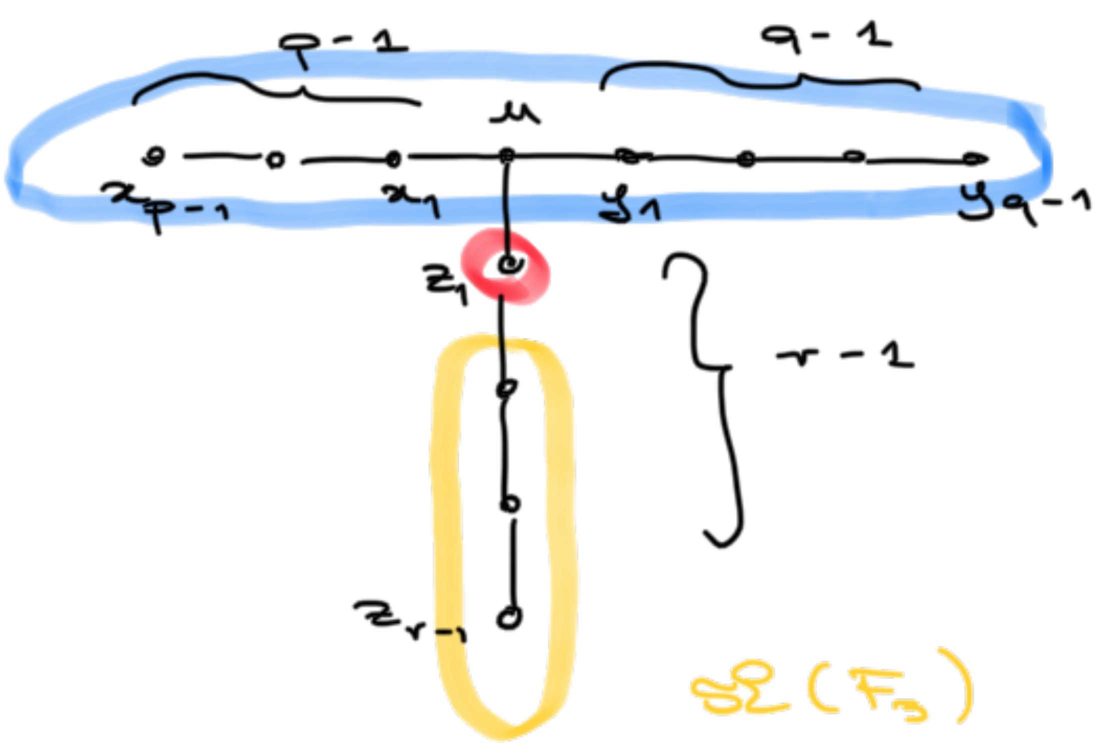
$\rightarrow$ can construct a generic ring, however in most cases $\hat{R}_{gen}$ is not Noetherian.

Weyman: associate to $0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$

$r_i = rk(d_i)$

a T_{pqr} - graph $\quad (p,q,r) = (r_1+1, r_2-1, r_3+1)$

$sl(F_2)$



$sl(F_3)$

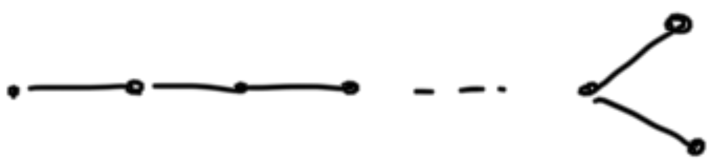
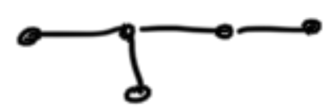
There exists a specific generic ring $\hat{R}_{gen}$ which has a multiplicity free action of the Lie algebra $sl(F_2) \times sl(F_0) \times g(T_{pqr})$

ADE cases

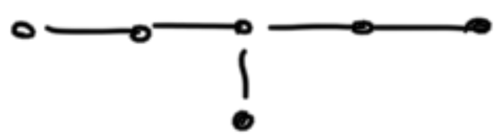
$\hat{R}_{gen}$ is Noetherian $\Leftrightarrow$ T_{pqr} is one of the ADE groups.

Resolutions of cyclic modules R/I:

A_n  $(1, 3, n, n-2)$

D_n  $(1, n, n, 1)$ 

$(1, 4, n, n-3)$ 

E_6  $(1, 5, 6, 2)$

E_7  $(1, 6, 7, 2)$

$(1, 5, 7, 3)$

E_8  $(1, 7, 8, 2)$

$(1, 5, 8, 4)$

Linkage

Assume $0 \to F_3 \to F_2 \to F_1 \to R \to R/I$

If I perfect, $(x_1, x_2, x_3) \subset I$ a regular sequence,

$J = (x_1, x_2, x_3) : I$ is also perfect.

(I and J are "linked".)

Moreover,

$$
\begin{array}{ccccccccccccc}
\mathbb{F}_\bullet : & 0 & \to & F_3 & \to & F_2 & \to & F_1 & \to & R & \to & R/I \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{K}_\bullet[x_1, x_2, x_3] : & 0 & \to & R & \to & \Lambda^2 R^3 & \to & R^3 & \to & R & \to & R/(x_1, x_2, x_3)
\end{array}
$$

Then the dual of the mapping cone resolves R/J.

LICCI ideals (Linkage Class of a Complete Intersection)

Conj: (Christensen - Veliche - Weyman)

(f_0, f_1, f_2, f_3) is Dynkin $\Leftrightarrow$
every perfect ideal with R/I - resolution of this format is licci.

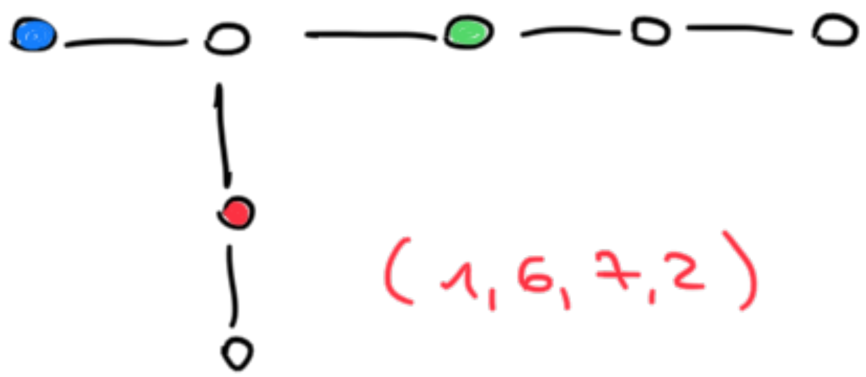
Note: If the format is not Dynkin, then we have non-licci ideals with resolution of that format.

Ex: $(1, n, n, 1)$ $(1, 4, n, n-3)$ A. Brown

$\uparrow$ Gorenstein $\uparrow$ almost complete intersection

Can find a pair of ideals inside the space of skew-symmetric matrices which have resolutions of these formats.

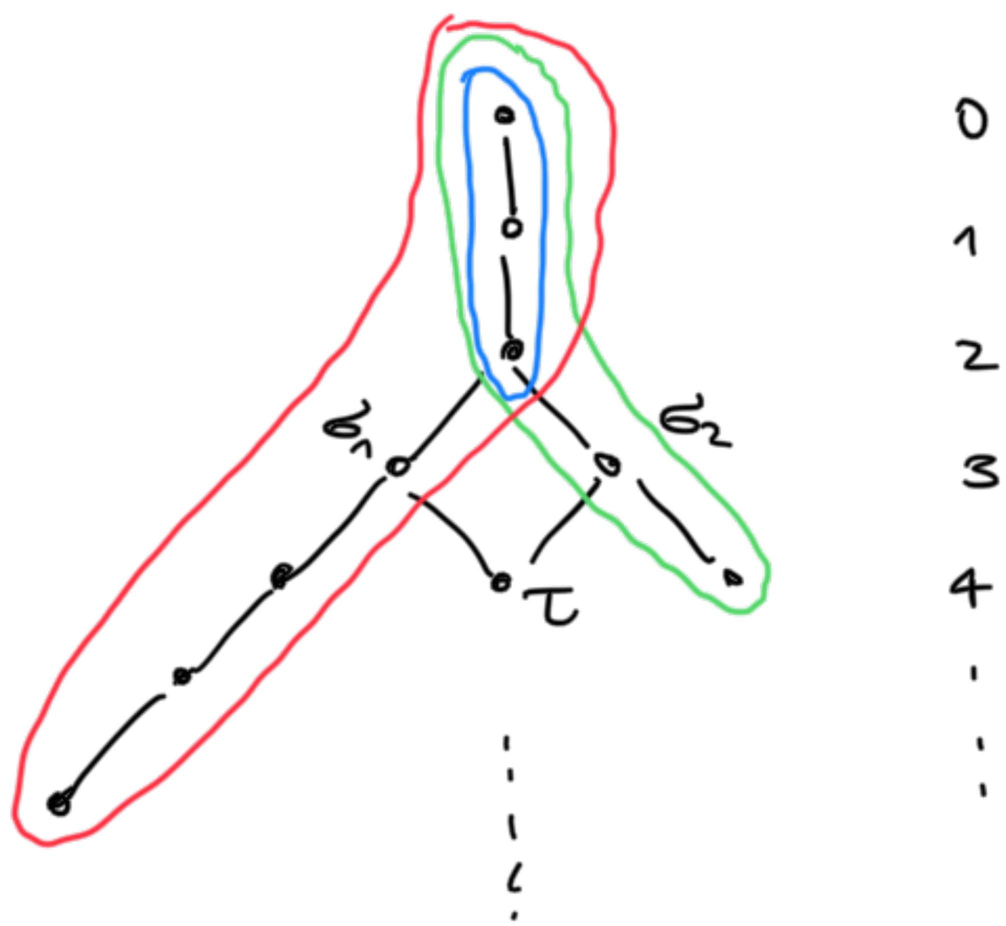
Sam – Weyman:



$(1,6,7,2)$ $(1,5,7,3)$

Schubert varieties in the homogeneous space G/P

Bruhat graph:



opposite Schubert varieties

$\updownarrow$

Plücker coordinates

Big opposite cell Y

$X_{\sigma_1} \cap Y = Y_{\sigma_1}$, $X_{\sigma_2} \cap Y = Y_{\sigma_2}$.

Thm: For all Dynkin formats D_n, E_6, E_7, E_8 the resolution of $K[Y_{\sigma_i}]$ as a $K[Y]$-module has the required Dynkin format.

Conjecture: These Schubert varieties have generic resolutions for perfect ideals of codim 3 with Dynkin formats (up to deformation)

(S, J) is a deformation of (R, I) if

$\exists$ a regular sequence $\underline{a} = (a_1 \text{———} a_r)$ on S and S/J st

$$R = S/(\underline{a}) \quad \text{and} \quad I = (\underline{a}) + J.$$

Question of Peskine–Szpiro: Either all these Schubert varieties are generic or QPS is false.

Ongoing projects

1. Resolution of Schubert varieties in the non-minuscule case

2. Generalization to extended Dynkin formats, e.g. $(1,6,8,3)$ $\hat{E}_7$

3. Explicit description of resolutions of length 3 in open subsets of the spectrum of the generic ring (Guerrieri - Weyman '2020, F. - Guerrieri : module formats in progress)

4. Gorenstein ideals of codim 4 ....

Thank you !