Quantum K-groups on flag manifolds

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In the beginning...

G : semi-simple, simply connected algebraic group over $\mathbb{C}$
$\rightsquigarrow G[\![z]\!] := G(\mathbb{C}[\![z]\!]) \subset G(\!(z)\!) := G(\mathbb{C}(\!(z)\!))$
$T \subset B$: maximal torus of $G \subset$ maximal solvable subgroup of G
$\rightsquigarrow X := G/B$: flag variety, $\mathrm{Gr}_G := G(\!(z)\!)/G[\![z]\!]$: affine Grassmannian

- We have
$$\mathrm{Gr}_G \sim \mathtt{Map}(S^1, K)/K,$$
where $K \subset G$ is the maximal compact subgroup of G $(\mathrm{Lie}\, K \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Lie}\, G)$;
- This is an "affine" analogue of the isomorphism
$$X = G/B \cong K/(T \cap K).$$
We have more obvious "affine" analogue of the above
$$\mathrm{Gr}_G \not\sim \mathtt{Map}(S^1, K/(T \cap K)).$$
- The two spaces are *different*, but number theorists observed some connection (spherical and Whittaker functions) around 1970.

Algebraic analogue of $\mathtt{Map}(S^1, \bullet)$

A C^∞-map $f : S^1 \to \mathbb{R}$ admits the Fourier expansion, that (sometimes) yields a complex analytic function near $S^1 \subset \mathbb{C}$. Hence, an algebraic analogue of $\mathtt{Map}(S^1, \bullet)$ is a rational map with its domain $\mathbb{C}$, or $\mathbb{P}^1$.

For an algebraic variety $\mathfrak{X}$, we set

$$Map(\mathbb{P}^1, \mathfrak{X}) = \{f : \mathbb{P}^1 \to \mathfrak{X} \mid \text{morphism of algebraic varieties}\}.$$

We have a natural decomposition

$$Map(\mathbb{P}^1, \mathfrak{X}) = \bigsqcup_{\beta \in H_2(\mathfrak{X}, \mathbb{Z})} Map_\beta(\mathbb{P}^1, \mathfrak{X}), \quad \text{where}$$

$$Map_\beta(\mathbb{P}^1, \mathfrak{X}) = \{f \in Map(\mathbb{P}^1, \mathfrak{X}) \mid f_*[\mathbb{P}^1] = \beta \in H_2(\mathfrak{X}, \mathbb{Z})\}.$$

We have two compactifications

$$\mathfrak{X}_\beta \supset Map_\beta(\mathbb{P}^1, \mathfrak{X}) \subset QMap_\beta(\mathbb{P}^1, \mathfrak{X}),$$

constructed by Kontsevich and Drinfeld, respectively.

Quantum cohomology and K-group of X

We have $H_2(X, \mathbb{Z}) \cong Q^\vee$ (the coroot lattice of G), and the condition $Map_\beta(\mathbb{P}^1, X) \neq \emptyset$ is equivalent to $\beta \in Q_+^\vee$ (the positive coroot span of G).

The (small T-equivariant) quantum cohomology group $qH_T^\bullet(X)$ and K-group $qK_T(X)$ are defined to be

$$qH_T^\bullet(X) := H_T^\bullet(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}Q_+^\vee, \quad qK_T(X) := K_T(X) \otimes_{\mathbb{C}} \mathbb{C}Q_+^\vee, \quad \text{(as vector space)}$$

with their multiplications $\star$ defined by using $X_{\beta,3}$ (a variant of X_β such that the genus zero curve has three marked points).

They are commutative rings with unit, and we have

$$a \star b \equiv a \cdot b \mod (Q^\beta \in Q_+^\vee \mid \beta \in Q_+^\vee \setminus \{0\}),$$

where $a, b \in qH_T^\bullet(X)$ or $a, b \in qK_T(X)$ and $\cdot$ are usual multiplications.

These are rings related to $\mathtt{Map}(S^1, K/(T \cap K))$.

Homology and K-group of Gr_G

$\mathsf{ev}_0 : G[\![z]\!] \to G$: setting $z = 0$, $\mathbf{I} := \mathsf{ev}_0^{-1}(B) \subset G[\![z]\!]$: Iwahori subgroup
$\rightsquigarrow$ $\exists$ bijection $\mathbf{I}\backslash\mathrm{Gr}_G \ni \mathrm{Gr}^\gamma \leftrightarrow \gamma \in Q^\vee$.

Since $\mathrm{Gr}_G = \bigcup_\gamma \overline{\mathrm{Gr}^\gamma_G}$, we form

$$H^T_\bullet(\mathrm{Gr}_G, \mathbb{C}) := \varinjlim H^T_\bullet(\overline{\mathrm{Gr}^\gamma_G}, \mathbb{C}), \quad K_T(\mathrm{Gr}_G) := \varinjlim K_T(\overline{\mathrm{Gr}^\gamma_G}).$$

We have their subspaces $H^G_\bullet(\mathrm{Gr}_G, \mathbb{C})$ and $K_G(\mathrm{Gr}_G)$ spanned by G-stable classes.

In case the two classes a, b in $H^G_\bullet(\mathrm{Gr}_G, \mathbb{C})$ or in $K_G(\mathrm{Gr}_G)$ admits G-action, we can define $a \odot b$ by using the maps

$$(\mathrm{Gr}_G)^2 \leftarrow G(\!(z)\!) \times \mathrm{Gr}_G \to \mathrm{Gr}_G.$$

This product extends to

$$H^T_\bullet(\mathrm{Gr}_G, \mathbb{C}) \cong H^\bullet_T(\mathrm{pt}) \otimes_{H^\bullet_G(\mathrm{pt})} H^G_\bullet(\mathrm{Gr}_G, \mathbb{C})$$
$$K_T(\mathrm{Gr}_G) \cong K_T(\mathrm{pt}) \otimes_{K_G(\mathrm{pt})} K_G(\mathrm{Gr}_G)$$

by linearity.

The "Pontrjyagin product" of Gr_G

Our product $\odot$ in $H^G_\bullet(\mathrm{Gr}_G, \mathbb{C})$ and $K_G(\mathrm{Gr}_G)$ are commutative!

Thus, $\odot$ are also commutative on $H^T_\bullet(\mathrm{Gr}_G, \mathbb{C})$ and $K_T(\mathrm{Gr}_G)$. In fact, $\odot$ is a unique multiplication structure that commutes with the localization with respect to $T \times \mathbb{G}_m$ ($\mathbb{G}_m$ acts on z by the scalar; loop rotation), and it commutes with T-character twists.

This property distinguishes the Pontrjyagin product (Lam-Schilling-Shimozono), that is the natural product structure arising from $\mathtt{Map}(S^1, K)/K$.
$\rightsquigarrow$ the convolution products and the Pontrjyagin product are the "same" in this sense (but may not in other sense).

We can also through-in the above $\mathbb{G}_m$ to consider $K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)$. This product is no longer commutative, but we still call the ring structure provided by

$$K_{T\times\mathbb{G}_m}(\mathrm{Gr}_G) \cong K_T(\mathrm{pt}) \otimes_{K_G(\mathrm{pt})} K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)$$

the Pontrjagin product by the characterization similar to the above.

Localizations of $H_\bullet^T(\mathrm{Gr}_G, \mathbb{C})$ and $K_T(\mathrm{Gr}_G)$

The subsets

$$H_\bullet^G(\mathrm{Gr}_G, \mathbb{C}) \subset H_\bullet^T(\mathrm{Gr}_G, \mathbb{C}) \quad \text{and} \quad K_G(\mathrm{Gr}_G) \subset K_T(\mathrm{Gr}_G)$$

contains the structure sheaves on $G[\![z]\!]$-orbit closures on Gr_G, that contains an open dense $\mathbf{I}$-orbit of the form

$$\mathrm{Gr}^\gamma \quad \gamma \in Q^\vee_< := \{\gamma \in Q^\vee \mid \gamma \text{ is anti-dominant}\}.$$

Note that $Q^\vee_+ \cap Q^\vee_< = \{0\}$. The fundamental classes or structure sheaves $\mathcal{O}^\gamma$ on $\overline{\mathrm{Gr}_G^\gamma}$ ($\gamma \in Q^\vee_<$) satisfies

$$\mathcal{O}^\gamma \odot \mathcal{O}^{\gamma'} = \mathcal{O}^{\gamma+\gamma'}.$$

We use these multiplicative system to localize $H_\bullet^T(\mathrm{Gr}_G, \mathbb{C})$, $K_T(\mathrm{Gr}_G)$ and $K_{T\times\mathbb{G}_m}(\mathrm{Gr}_G)$ to obtain $H_\bullet^T(\mathrm{Gr}_G, \mathbb{C})_{\mathrm{loc}}$, $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$ and $K_{T\times\mathbb{G}_m}(\mathrm{Gr}_G)_{\mathrm{loc}}$ (the latter is non-commutative localization).

Peterson isomorphism in homology

A relation between rings coming from

$$\mathtt{Map}(S^1, K/(T \cap K)) \quad \text{and} \quad \mathtt{Map}(S^1, K)/K$$

is observed in 1990s (cf. the works of Givental and Kim).

Theorem (Peterson 1997, Lam-Shimozono 2010)

We have an isomorphism of algebras (up to localizations on the both sides)

$$qH^\bullet_T(X)_{\mathrm{loc}} \cong H^T_\bullet(\mathrm{Gr}_G, \mathbb{C})_{\mathrm{loc}}.$$

1. The localization of $qH^\bullet_T(X)_{\mathrm{loc}}$ is $\mathbb{C}Q \otimes_{\mathbb{C}Q^\vee_+}$;
2. We have $qH^\bullet_T(X) \cap H^T_\bullet(\mathrm{Gr}_G, \mathbb{C}) = H^\bullet_T(\mathrm{pt})[\mathcal{O}^0]$. In other words, the genuine common part is basically just 1;
3. This isomorphism respects bases, and hence not a mere isomorphism;
4. The quantum cohomology of the partial flag manifolds are also read-out by examining this isomorphism.

Lam-Li-Mihalcea-Shimozono's conjecture

Let W be the Weyl group of G. The group W acts on P and $Q^\vee$. The set W parametrizes the B-orbits of X as $w \mapsto X_w^\circ$ such that

$$\dim X_w = \ell(w),$$

where ℓ is the length function of W. We set $X_w := \overline{X_w^\circ}$ (the Schubert variety).

Conjecture (Lam-Li-Mihalcea-Shimozono 2018)

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\mathtt{Pet} : qK_T(X)_{\mathrm{loc}} \cong K_T(\mathrm{Gr}_G)_{\mathrm{loc}},$$

where we have

$$\mathtt{Pet}(Q^\beta) = \mathcal{O}^\beta \quad (\beta \in Q^\vee) \quad \mathtt{Pet}(\mathcal{O}_{X_w}) = \mathcal{O}^{w\gamma}(\mathcal{O}^\gamma)^{-1},$$

where $\gamma \in Q^\vee$ is strictly dominant. Here we have extended the range of β by expressing $\beta = \beta_1 - \beta_2$ for $\beta_1, \beta_2 \in Q^\vee_+$ or $\beta_1, \beta_2 \in Q^\vee_<$ and introduce

$$Q^\beta = Q^{\beta_1}(Q^{\beta_2})^{-1} \qquad \mathcal{O}^\beta = \mathcal{O}^{\beta_1}(\mathcal{O}^{\beta_2})^{-1}.$$

Remarks on LLMS conjecture

Conjecture (Lam-Li-Mihalcea-Shimozono 2018)

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$$Q^\beta = Q^{\beta_1}(Q^{\beta_2})^{-1} \qquad \mathcal{O}^\beta = \mathcal{O}^{\beta_1}(\mathcal{O}^{\beta_2})^{-1}.$$

1. The element $\mathcal{O}^{w\gamma}(\mathcal{O}^\gamma)^{-1}$ is independent of the choice of γ, but we cannot set it to 0 as $w0 = 0$ and hence $\mathcal{O}^{w0}(\mathcal{O}^0)^{-1} = \mathcal{O}^0(\mathcal{O}^0)^{-1} = 1 (= \mathcal{O}^0)$;
2. We have $\mathtt{Pet}(\mathcal{O}_{X_w} Q^\beta) = \mathcal{O}^{w\gamma}(\mathcal{O}^\gamma)^{-1}\mathcal{O}^\beta$;

Our main idea in a proof of LLMS conjecture

Introduce the third object, that can be understood as another instance of $\mathtt{Map}(S^1, K/(T \cap K))$. Let $N := [B, B]$. We define the semi-infinite flag manifolds as:

$$\mathbf{Q}_G^{\mathrm{rat}} := G(\!(z)\!)/(T \cdot N(\!(z)\!))$$

This is a version of $\mathtt{Map}(\mathbf{D}^\times, G/N)/T$, where $\mathbf{D}^\times = \mathrm{Spec}\,\mathbb{C}(\!(z)\!)$ is a version of the "germ of punctured open subset", another algebraic avatar of S^1.

We have the notion of equivariant K-group $K_{T \times \mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$. Using this, our formulation of the above isomorphism becomes:

Theorem (arXiv:1805.01718)

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\begin{array}{ccc} & K_T(\mathbf{Q}_G^{\mathrm{rat}}) & \\ \overset{\Phi}{\nearrow} & & \overset{\Psi}{\nwarrow} \\ K_T(\mathrm{Gr}_G)_{\mathrm{loc}} & \longrightarrow & qK_T(X)_{\mathrm{loc}} \end{array}.$$

A non-commutative version

Introducing the $\mathbb{G}_m$-action that acts on z for Gr_G and $\mathbf{Q}_G^{\mathrm{rat}}$, we have:

Theorem (arXiv:1805.01718 + arXiv:2008.01310)

We have an isomorphism of based algebras (up to localizations on the both sides)

$$\begin{array}{ccc} & K_{T \times \mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}}) & \\ {}^{\Phi}\nearrow & & \nwarrow^{\Psi} \\ K_{T \times \mathbb{G}_m}(\mathrm{Gr}_G)_{\mathrm{loc}} & \longrightarrow & qK_{T \times \mathbb{G}_m}(X)_{\mathrm{loc}} \end{array}.$$

What is $\mathbb{G}_m$ in $qK_{T \times \mathbb{G}_m}(X)$? This is the marking of one $\mathbb{P}^1$ in the genus zero curve ($=$ union of several copies of $\mathbb{P}^1$). It is an ingredient to define the quantum difference equation (QDE).

In particular, we have some non-commutative algebra structure on $qK_{T \times \mathbb{G}_m}(X)_{\mathrm{loc}}$. This is a consequence of the fact that the QDE appearing here is governed by the q-Heisenberg algebra (with q degree one character of $\mathbb{G}_m$).

A commutative diagram

In particular, factoring out a component of q-Heisenberg algebra yields:

Theorem (arXiv:2008.01310)

We have a commutative diagram of rings

$$\begin{array}{ccc} K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)_{\mathrm{loc}} & \longrightarrow & qK_{G\times\mathbb{G}_m}(X)_{\mathrm{loc}} \\ \big\downarrow & & \big\uparrow \\ K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L)_{\mathrm{loc}} & \hookleftarrow & qK_{[L,L]\times\mathbb{G}_m}(X^L)_{\mathrm{loc}} \end{array} ,$$

where $T \subset L \subset G$ is a semi-simple Levi subgroup, and X^L is the flag variety of L.

The q-Heisenberg relation is

$$(\star\mathcal{O}_X(-\varpi_i))Q^{\alpha_j^\vee} = q^{\delta_{ij}}Q^{\alpha_j^\vee}(\star\mathcal{O}_X(-\varpi_i)),$$

where ϖ_i is a fundamental weight and $\alpha_j^\vee$ is a simple coroot associated to G. The ones corresponding to L remain as is, and otherwise $(\star\mathcal{O}_X(-\varpi_i))$ reduces to a T-character twist (but still non-commutative!).

Note that the LHS is larger than the RHS in this case.

A commutative diagram of commutative rings

By setting $q = 1$ and extend the scalar, we find

Theorem (arXiv:2008.01310)

We have a commutative diagram of commutative rings

$$\begin{array}{ccc} K_T(\mathrm{Gr}_G)_{\mathrm{loc}} & \longrightarrow & qK_T(X)_{\mathrm{loc}} \\ \downarrow & & \downarrow\!\!\downarrow \\ K_T(\mathrm{Gr}_L)_{\mathrm{loc}} & \twoheadrightarrow & qK_T(X^L)_{\mathrm{loc}} \end{array} .$$

We can also drop loc in the LHS as conjectured by Finkelberg-Tsymbaliuk. Explicitly, we have

$$K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G) \hookrightarrow K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L) \hookrightarrow K_{T\times\mathbb{G}_m}(\mathrm{Gr}_T) \equiv q\text{-}\mathrm{Heis}_T.$$

The image of $K_G(\mathrm{pt}) \subset K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)$ in $K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L)$ is not contained in $K_L(\mathrm{pt}) \subset K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L)$!

Another commutative diagram

For each standard parabolic $B \subset P \subset G$, we set $X_P := G/P$. The Peterson isomorphism for $qH(X)$ describes $qH(X_P)$ in terms of $H(\mathrm{Gr}_G)$.

In our case, we have

Theorem (arXiv:1906.09343)

We have a commutative diagram

$$\begin{array}{ccc} K_{T\times\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}}) & \longrightarrow & qK_T(X)_{\mathrm{loc}} \\ \downarrow & & \downarrow \\ K_{T\times\mathbb{G}_m}(\mathbf{Q}_{G,P}^{\mathrm{rat}}) & \longrightarrow & qK_T(X_P)_{\mathrm{loc}} \end{array},$$

where

$$\mathbf{Q}_{G,P}^{\mathrm{rat}} := G((z))/(T\cdot[P,P]((z)))$$

is the parabolic version of $\mathbf{Q}_G^{\mathrm{rat}}$. The vertical arrow in the RHS is a ring map.

We can restrict the LHS to drop loc in the RHS.

Another commutative diagram

Let Π_P be the set of simple roots of Levi of P. We have

$$qK_{T\times\mathbb{G}_m}(X_P) = \mathbb{C}[q^{\pm 1}] \otimes K_T(X_P)[Q^\beta \mid \beta \in \mathbb{Z}_{\geq 0}\Pi_P].$$

Theorem (arXiv:1906.09343)

$$\begin{array}{ccccc} K_{T\times\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}}) & \longrightarrow & qK_{T\times\mathbb{G}_m}(X)_{\mathrm{loc}} & \longleftarrow & qK_{T\times\mathbb{G}_m}(X) \\ \downarrow & & \downarrow & & \downarrow \\ K_{T\times\mathbb{G}_m}(\mathbf{Q}_{G,P}^{\mathrm{rat}}) & \longrightarrow & qK_{T\times\mathbb{G}_m}(X_P)_{\mathrm{loc}} & \longleftarrow & qK_{T\times\mathbb{G}_m}(X_P) \end{array},$$

where the map

$$qK_{T\times\mathbb{G}_m}(X) \ni [\mathcal{O}_{X_w}]Q^\beta \mapsto [\mathcal{O}_{X_{P,w}}]Q^{[\beta]} \in qK_{T\times\mathbb{G}_m}(X_P) \qquad w \in W, \beta \in Q_+^\vee$$

is defined so that $X_{P,w}$ is the image of X_w under the projection map $X \to X_P$, and

$$\beta = \sum_i a_i\alpha_i^\vee \mapsto [\beta] = \sum_{\alpha_i \in \Pi_P} a_i\alpha_i^\vee .$$

Why this has a chance to hold?

The ring map

$$qK_T(X) \to qK_T(X_G) \equiv qK_T(\mathrm{pt}) = K_T(\mathrm{pt})$$

is proved previously by Buch-Chung-Li-Mihalcea. Let us explain this for $G = SL(2)$ from our perspective. We have

$$\mathbf{Q}_G^{\mathrm{rat}} \supset \mathbf{Q}_G = \mathbb{P}(\mathbb{C}^2[\![z]\!]) = \mathbb{P}^\infty, \mathbf{Q}_{G,G}^{\mathrm{rat}} = \mathrm{pt}.$$

According to our claim, we have an isomorphism

$$K(\mathbb{P}^\infty) \to qK(X) \equiv qK(\mathbb{P}^1)$$

that sends $\mathcal{O}_{\mathbf{Q}_G}(n)$ to $\mathcal{O}_X(n)$ for $n = 0, -1$. We have

$$[\mathcal{O}_X] = [\mathcal{O}_{X_e}] \mapsto [\mathcal{O}_{\mathrm{pt}}] = 1 \in K_T(\mathrm{pt}).$$

$$[\mathcal{O}_X(-1)] = [\mathcal{O}_X] - [\mathcal{O}_{X_s}] \mapsto 1 - 1 = 0.$$

By the quantum multiplication rule $[\mathcal{O}_{X_s}] \star [\mathcal{O}_{X_s}] = [\mathcal{O}_{X_e}] Q^{\alpha^\vee} \mapsto [\mathcal{O}_{\mathrm{pt}}] = 1$ ($Q^{\alpha^\vee} = 1$), we have

$$[\mathcal{O}_X(-1)]^{\star n} \mapsto 0 \qquad n > 0.$$

The corresponding fact in $K(\mathbb{P}^\infty)$ is

$$H^\bullet(\mathbb{P}^\infty, \mathcal{O}(-n)) = 0 \qquad n > 0.$$

Some disclaimers on $\mathrm{Q}^{\mathrm{rat}}$

I should have explained about $\mathbf{Q}^{\mathrm{rat}}$ in a bit more systematic way here, but today I choose just to make some comments.

- The geomery of $\mathbf{Q}_G^{\mathrm{rat}}$ is closely related to that of $QMap_\beta(\mathbb{P}^1, X)$, but the implication of statements are usually only $\mathbf{Q}_G^{\mathrm{rat}} \Rightarrow QMap_\beta(\mathbb{P}^1, X)$. The main reason is that the coordinate ring of $\bigcup_\beta QMap_\beta(\mathbb{P}^1, X)$ is the completion of $\mathbf{Q}_G^{\mathrm{rat}}$, and this completion is exact but forgets some information;
- Proofs of some key properties (e.g. Frobenius splitting) of $\mathbf{Q}_G^{\mathrm{rat}}$ requires completely new ideas as known proof strategies for usual flag varieties (including Kac-Moody cases) break down for $\mathbf{Q}_G^{\mathrm{rat}}$:
  - K. arXiv:1810.07106, *Forum of Math., Pi* **9** (2021).
- I am now writing a survey paper related to the ideas of these and related works, that is supposed to be available in 2022.