Stable envelopes and motivic Chern classes for cotangent varieties

Everything over $\mathbb{C}$

parabolic

maximal torus

affine reductive group

$$A \subset P \subset G$$

$A \curvearrowright G/P =: M$

$A \times \mathbb{C}^* \curvearrowright T^*G/P =: X$

Fix general enough $\sigma : \mathbb{C}^* \to A$

$e \in M^A \rightsquigarrow M_e^+ = \{ x \in M \mid \lim_{t \to 0} \sigma(t) \cdot x = e \}$

$e \in X^A \rightsquigarrow X_e^+ = \{ x \in X \mid \lim_{t \to 0} \sigma(t) x = e \} = \nu^*(M_e^+ \hookrightarrow M)$

$[\overline{X_e^+}]$ - cotangent Schubert calculus

$$[\overline{X_e^+}] \in K^{A \times \mathbb{C}^*}(X) \simeq K^{A \times \mathbb{C}^*}(M)$$

stable envelope

motivic Chern class

Stable envelopes for $X = T^*G/P$

$s \in \mathrm{Pic}(X) \otimes \mathbb{Q}$

$$e \in X^A \rightsquigarrow \mathrm{stab}^s(e) \in K^{A \times \mathbb{C}^*}(X)$$

1) $\mathrm{supp}(\mathrm{stab}^s(e)) \subset \bigsqcup_{e' \leq e} X^+_{e'}$

2) $\mathrm{stab}^s(e)|_e = [\overline{X^+_e}] \cdot \mathcal{L}_e$

3) for $e' < e$

$N^A(\mathrm{stab}^s(e)|_{e'}) \subset$ some given polytope

$$\begin{array}{ccc} K^{A \times \mathbb{C}^*}(e') & \simeq & K^{\mathbb{C}^*}(e')[t_1^{\pm}, \dots, t_{\dim A}^{\pm}] \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ a & = & \sum a_I t^I \end{array}$$

$$N^A(a) = \mathrm{conv}\{ I \in \mathbb{Z}^{\dim A} \mid a_I \neq 0 \}$$

Motivic Chern class

How to generalize $c^*(TY)$ to singular varieties?

MacPherson : c_{SM} class

$$c_{SM} : F(-) \to H_*(-)$$

[BSY] mC class

[AMSS]; [FRW] equivariant mC class

T-equivariant map of varieties $f : Z \to Y$

$$\Big\downarrow mC_y^T$$

$$\text{class } mC_y^T(f) \in K^T(Y)[y]$$

1) Y-smooth

$$mC_y^T(id_Y) = \lambda_y(T^*Y)$$

2) $g : Y \to W$ proper

$$mC_y^T(Z \xrightarrow{f} Y \xrightarrow{g} W) = g_* \, mC_y^T(Z \xrightarrow{f} Y)$$

3) For open $U \subset Z$

$$mC_y^T(Z \xrightarrow{f} Y) = mC_y^T(U \to Y) + mC_y^T(Z/U \to Y)$$

y - deformation

$$Z^{o} \overset{\text{open}}{\subset} Z \overset{\text{closed}}{\subset} Y$$

$$\begin{tikzcd}
K^T(Y)[y] \arrow[r, "y=0"] & K^T(Y) \\
mC_y^T(Z \subset Y) \arrow[r, "y=0"] \arrow[u, phantom, sloped, "\in"] & {[Z]} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$\begin{tikzcd}
K^T(Z_0)[y] \arrow[r, "y=0"] & K^T(Z_0) \\
mC_y^T(Z^0 \subset Y)_{|Z^0} \arrow[r, "y=0"] \arrow[u, phantom, sloped, "\in"] & {[Z]_{|Z^0}} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

Comparison of stab and mC

[AMSS], [FR], [RV] : for T^*G/P

Description of cohomological version of stab as c_{SM} class

$X = T^*G/P$; $M = G/P$ $\quad \pi : X \to M$

[AMSS] For T^*G/B, s-small antiample

$$stab^s(e) = \pm h^c \cdot \pi^* \, {}_mC_{-h}^{A\times\mathbb{C}^*}(M_e^+ \to M)$$

where $h : A \times \mathbb{C}^* \to \mathbb{C}^*$, $h \in K^{A\times\mathbb{C}^*}(pt)$

[FRW]; [RTV]

The same formula holds for T^*GL_n/P

a) For T^*G/P, s-small antiample

$$\mathrm{stab}^s(e) = \pm \hbar^{c} \cdot \pi^* \; {}_m C^{A\times\mathbb{C}^*}_{-\hbar}(M_e^+ \to M)$$

b) For $X = T^*M$; M-smooth, projective

$A \curvearrowright M$, M^A discrete

LHS satisfies Newton and normalization axioms

for s-small antiample

c) Moreover if BB decomposition of M is nice enough

then LHS satisfies support axiom

Arbitrary slope and twisted mC

$s \in Pic(M) \otimes \mathbb{Q} \rightsquigarrow \Delta_{s,e} \in \mathbb{Q}Weil(\overline{M_e^+})$

$$\begin{tikzcd} & & \widetilde{M}_e \arrow[d, "p"] & & & T^*G/P = X \arrow[d, "\pi"] \\ M_e^+ \arrow[urr] \arrow[rr] & & \overline{M_e^+} \arrow[r, "i"] & M & & G/P = M \end{tikzcd}$$

$$mC_y(M_e^+ \to \overline{M_e^+}; s) = p_*\left(mC_y(M_e^+ \to \widetilde{M}_e)\, \mathcal{O}(\lceil p^*\Delta_{s,e} \rceil)\right)$$

For s - small antiample

$$mC_y(M_e^+ \to \overline{M_e^+}; s) = mC_y(M_e^+ \to \overline{M_e^+})$$

For T^*G/P, s - arbitrary

$$stab^s(e) = \pm h^{c} \cdot \pi^* i_* mC_{-h}^{A \times \mathbb{C}^*}(M_e^+ \to \overline{M_e}; s)$$