VMRT of wonderful compactifications of symmetric spaces

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VMRT = Variety of Minimal Rational Tangents.

1) VMRT
2) Wonderful comp° of symmetric spaces.
3) Restricted root systems.
4) Results + idea of proof.

1) VMRT.

- X proj. smooth $/\mathbb{C}$ uniruled (covered by rat curves).
- Fix $x \in X$ "general".

Def: $\beta \in H_2(X,\mathbb{Z})$.

(1) β is **covering** if $ev: M_{0,1}(X,\beta) \longrightarrow X$, $(f,p_1) \longmapsto f(p_1)$

$$M_{0,1}(X,\beta) = \left\{ f: \mathbb{P}^1 \to X \;\middle|\; \begin{array}{l} p_1 \in \mathbb{P}^1 \\ f_*[\mathbb{P}^1] = \beta \end{array} \right\} \Big/ \sim$$

is **dominant**.

(2) β is **minimal** if β is covering

and $K_{x,\beta} = ev^{-1}(x) = \left\{ f: \mathbb{P}^1 \to X \;\middle|\; \begin{array}{l} f(p_1) = x \\ f_*[\mathbb{P}^1] = \beta \end{array} \right\}$.

$K_{x,\beta}$ is **proper**.

Ex: (1) $X = \mathbb{P}^n$ $\quad \beta \in H_2(X,\mathbb{Z}) = \mathbb{Z}$.

$\beta > 0 \iff \beta$ covering.

$\beta = 1 \iff \beta$ minimal.

(ii) $X = \mathrm{Grass}(p,n) = G/P$ $\quad P$ maximal, $\mathrm{Pic}(X) = \mathbb{Z}$.

$\beta \in H_2(X,\mathbb{Z}) = \mathbb{Z}$.

β covering $\iff \beta > 0$

β minimal $\iff \beta = 1$.

(iii) $X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad \beta \in \mathbb{Z}^2$.

$(a,b) = \beta$ covering $\Leftrightarrow$ $a \geq 0$, $b \geq 0$, $a+b > 0$.

β minimal $\Leftrightarrow$ $\beta = (1,0)$ or $(0,1)$.

(iv) $X = G/B$. $\quad \beta \in H_2(X,\mathbb{Z}) =$ coroots.

β covering $\Leftrightarrow \beta = \sum a_i \alpha_i^\vee \quad a_i \geq 0$
$\beta \neq 0$.

β minimal $\Leftrightarrow$ $\beta = \alpha_i^\vee$ simple coroot.

(v) $X =$ cubic surface, lines are not covering.

<u>Def</u>: Let β minimal.

Consider $\tau : K_{x,\beta} \longrightarrow \mathbb{P}(T_x X)$.

$$\begin{pmatrix} f: \mathbb{P}^1 \to X \\ p_1 \mapsto x \end{pmatrix} \longmapsto [df(p_1)].$$

$\mathrm{VMRT}_\beta(X) =$ image of $\tau \subset \mathbb{P}(T_x X)$.

<u>Ex</u>: (1) $X = \mathbb{P}^n \rightsquigarrow \mathbb{P}^{n-1} \subset \mathbb{P}^{n-1} = \mathbb{P}(T_x X)$.

(2) $X = Gr(p,n)$ $\quad \mathrm{VMRT}(X) = \mathbb{P}^{p-1} \times \mathbb{P}^{n-p-1} \underset{\text{segre}}{\hookrightarrow} \mathbb{P}(V_p \otimes \mathbb{C}^n/V_p)$

$V_p \subset \mathbb{C}^n$ $\qquad \qquad$ $\mathbb{P}^{p-1} \times \mathbb{P}^{n-p-1} = \mathbb{P}V_p \times \mathbb{P}(\mathbb{C}^n/V_p)$

Fact: $X = G/P$ P maximal $\operatorname{Pic}(X) = \mathbb{Z}$.
VMRT(X) is given by lines, and almost always homogeneous under P.

2) Wonderful comp° of symm. spaces.

Def: Let G be reductive gp. and σ an gp. involution. Set $H = G^\sigma$

$$G/H = G/G^\sigma \text{ is a homo. sym. spa}$$

Ex: (1) $G = H \times H$ $\sigma(x,y) = (y,x)$ $G^\sigma = \mathrm{diag}(H)$.

$G/G^\sigma \simeq H$.

(2) $SL_n = G$, $H = SO_n$, $\sigma(M) = (M^t)^{-1}$.

(3) SL_{2n}, Sp_{2n}.

(4) $G = PGL_n$, $H = P(GL_p \times GL_{n-p})$.

σ = conj. with $\begin{array}{cc} \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right) & \begin{array}{l} p \\ n-p \end{array} \end{array}$.

Def: A wonderful comp° of G/H a hom. sym. space is a G-variety X, <u>smooth</u>, <u>projective</u>.

(1) X has an open G-orbit $\mathring{X} = G/H$.

(2) $X \setminus \mathring{X} = \bigcup_{i=1}^{r} X_i$ $\quad X_i$ G-stable SNC divisors.

(3) Any G-orbit is of the form

$$\mathcal{O}_I = \bigcap_{i \in I} X_i \setminus \left(\bigcup_{i \notin I} X_i \right) \quad \forall I \subset [1, r].$$

- r = rank of $X = \mathrm{rk}(X)$.
- open orbit $\mathring{X} = \mathcal{O}_I$ $\quad I = \emptyset$.
- $\exists$ unique closed orbit $Y = \mathcal{O}_I$ $\quad I = [1, r]$.
  $= G/P$.

<u>Examples:</u> (1) $H = PGL_2$ $\quad G = H \times H$

$H = H \times H / H$. $\quad X = \mathbb{P}(M_2(\mathbb{C}))$ is the wonderful comp°.

$\mathrm{rk}(X) = \mathrm{rk}(H) = 1$.

$X_1 \,\cancel{E} = \{\mathrm{rk}\, 1 \text{ matrices}\}$.
$= \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = X$.

(2) $H = PGL_n$, $G = H \times H$.

$$Z = \mathbb{P}(M_n(\mathbb{C})) \supset Z_{n-1} \supset \dots \supset Z_1$$

$$Z_i = \{\text{matrices of rk} \leq i\}.$$

X = wond. comp. = Blow-up $Z_1, Z_2, \dots, Z_{n-2}$.

(3) $G = PGL_n$. $H = P(GL_p \times GL_{n-p})$.

$$Z = Gr(p,n) \times Gr(n-p,n).$$

$$\supset Z_1 \supset \dots \supset Z_p.$$

$$Z_i = \{(V_p, V_{n-p}) \mid \dim(V_p \cap V_{n-p}) \geq i\}.$$

X = Blow-up of $Z_p, Z_{p-1}, \dots, Z_1$.

3) Rest. root systems:

Start with G and σ an involution.

- Choose a maximal torus $T \subset G$ st $\sigma(T) = T$.
- Choose T a σ-stable max torus st.

$$T_1 = \{t \in T \mid \sigma(t) = t^{-1}\}^0 \text{ is maximal.}$$

Look at the root system $R = (G,T)$.

σ acts on R.

- Choose simple roots $\Delta \subset R$ such that:

  - either $\sigma(\alpha) = \alpha$.
  - or $\sigma(\alpha) < 0$.

Set $\Delta_1 = \{\alpha \in \Delta \mid \sigma(\alpha) < 0\}$.

Prop: Δ_1 is a basis of a root system (maybe not reduced). R_σ

Def: This root system R_σ is the rest. root system.

Ex: (1) $G = H \times H \rightsquigarrow R_\sigma =$ root system of H.

(2) SL_n/SO_n, $SL_{2n}/Sp_{2n} \rightsquigarrow R_\sigma = A_{n-1}$.

(3) $PGL_n / P(GL_p \times GL_{n-p}) \rightsquigarrow R_\sigma = BC_p$.

This root system gives a lot of info. on the geom of X.

<u>Facts:</u> (X non exceptional).

$\mathrm{Pic}(X)$ = weight lattice of R_σ.

$H_2(X,\mathbb{Z})$ = coroot lattice of R_σ.

pairing curves/divisors = pairing weight / coroots.

nef cone = dominant cone of R_σ.

effective cone of divisors = cone of roots $\oplus \mathbb{N}\alpha_i$.

$X_i \Leftrightarrow \alpha_i$. simple roots.

<u>Prop:</u> There is a unique ~~...~~ minimal covering class $\beta = \Theta^\vee$ where Θ = highest root.

<u>Proof:</u> β covering $\Rightarrow$ any curve of class β meets $\overset{\circ}{X}$ $\Rightarrow$ not contained in any X_i.

$\beta \cdot X_i \geq 0$, $\Leftrightarrow \langle \beta, \alpha_i \rangle \geq 0$.

$\Rightarrow$ β is in the cone of dom. coweights.

and β minimal and a coroots.

$\rightsquigarrow$ there is a unique minimum $\Theta^\vee$.

4) <u>Results</u>:

Describe $VMRT(X) \subset \mathbb{P}(T_{[id]}X)$

$[id] \in G/H \subset X$.

<u>Cartan decomp</u>:

$\mathfrak{g} = \mathrm{Lie}(G)$.

$\mathfrak{h} = \mathrm{Lie}(H)$

Decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as H-rep.

$\mathfrak{h}$ = eigenspace of $+1$ for σ.

$\mathfrak{p}$ = eigenspace of -1 for σ.

$\mathbb{P}(T_x X) = \mathbb{P}(\mathfrak{p}) \circlearrowleft H$ action.

$\rightsquigarrow VMRT(X) \subset \mathbb{P}(T_x X)$ has an H-action.

<u>Ex</u>: (1). $G = H \times H$ $\quad \mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$.

$\mathfrak{h} = \mathfrak{h}$, $\mathfrak{p} = \mathfrak{h}$. adj rep. of $\mathfrak{h}$.

(2) $G = GL_n$, $H = P(GL_p \times GL_{n-p})$.

$$\mathfrak{g} = \begin{array}{c} \left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right) \\ \begin{array}{cc} p & n-p \end{array} \end{array} = \underbrace{\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)}_{\mathfrak{h}} \oplus \underbrace{\left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)}_{\mathfrak{p}}$$

$\mathfrak{p} = V \oplus V^\vee$ as H-rep. V irred.

$H = GL_p \times GL_{n-p}$ $\qquad V = \mathbb{C}^p \otimes \mathbb{C}^{n-p}$.

<u>Fact:</u> Any X wond. comp. is a product of wond. comp' of hom. sym. spaces of 3 types.

(1) gp type : $\mathfrak{p}$ irred.

(2) case where $\mathfrak{p} = V \oplus V^\vee$ $\quad V$ irred.
(X exceptional).

(3) $\mathfrak{p}$ irred. G is simple.

<u>Fact:</u> $\mathfrak{p}$ is either irred. or $\mathfrak{p} = V \oplus V^\vee$ $\quad V$ irred.

$\rightsquigarrow v \in \mathfrak{p}$ or $v, v' \in V, V^\vee$ high. weight vect.

Thm [BKP] X wonderful comp. assume R_σ is irred.

(1) If R_σ is not of type A then

$$\mathrm{VNRT}(X) = H\cdot[v] \subset \mathbb{P}(\mathfrak{g}) \quad \text{if } X \text{ is not } \mathrm{etq}.$$

$$\rightsquigarrow \quad \mathrm{VNRT}(X) \begin{matrix} \nearrow H\cdot[v] \\ \searrow H\cdot[v'] \end{matrix} \subset \mathbb{P}(\mathfrak{g}) = \mathbb{P}(V\oplus V^\vee).$$

(2) In type $R_\sigma = A_n$. the $\mathrm{VNRT}(X)$ has dimension 1 more than $H\cdot[v]$ but it is homogeneous under a bigger gp. (hom. under G).

(3) $v, v' \in \mathfrak{g} = \mathrm{Lie}(G)$.

Describe in which nilpotent orbit v, v' live:

- if $\sigma(\Theta) = -\Theta$ then $v \in \mathcal{O}_{\text{minimal}}$ the minimal nilpotent orbit.
- if $\sigma(\Theta) \neq -\Theta$ then $v \in \mathcal{O} = G\cdot[x_\alpha + x_\beta]$.

  α, β strongly orth. roots.

Idea of pf:

- $V \cap RT(x)$ is proper + H action (red sp).

$\Rightarrow$ B_H - fixed pt in $V \cap RT(x)$.

$\Rightarrow$ $\exists$ B_H - vector in $\mathbb{P}(\mathfrak{g})$ lying on $V \cap RT(x)$.

$\Rightarrow$ $[v] \in V \cap RT(x)$.

$\Rightarrow$ $H \cdot [v] \subseteq V \cap RT(x)$.

- Compute dimension.
- upper bound for $V \cap RT(x)$.
- Compute $\dim H \cdot [v]$.

$$\underbrace{H \cdot v}_{\text{Lagrangian sub. in } G \cdot x} \subset \underbrace{G \cdot v}_{\text{know.}} \subset \text{nilpotent orbit.}$$