Positroid varieties and Catalan numbers

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based on joint work with P. Galashin

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$$Gr(k,n) = \coprod_{I \in \binom{[n]}{k}} \Omega_I \qquad X_I = \overline{\Omega_I}$$

Grassmannian — Schubert cell — Schubert variety

- $\chi(Gr(k,n)) = \dim H^*(Gr(k,n)) = \binom{n}{k}$
- $\chi(\Omega_I) = \dim H^*(\Omega_I) = 1$
- $\chi(X_I) = \dim H^*(X_I) = \#\{J \mid J \geq I\}$

Positroid varieties = intersections of cyclically rotated Schubert varieties

(Knutson-L.-Speyer)

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in Gr(k,n) \qquad v_{i+n} := v_i$$

$$f_V : \mathbb{Z} \to \mathbb{Z}$$

$$f_V(i) = \min\{ j \geq i \mid v_i \in \text{span}(v_{i+1}, v_{i+2}, \ldots, v_j) \}$$

e.g.

$$V = \begin{array}{c} 1\ \ 2\ \ 3\ \ 4\ \ 5 \\ \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

$f(1) = 3$ $\quad f(3) = 5$ $\quad f(5) = 9$

$f(2) = 6$ $\quad f(4) = 7$

Prop f_V is a (k,n)-bounded affine permutation:

- (affine perm) $f(i+n) = f(i)+n$, f bijection
- (component k) $\sum_{i=1}^{n}(f(i)-i) = kn$
- (bounded) $i \le f(i) \le i+n$

$f_{k,n}(i) = i+k$

Defn open positroid variety

$$\mathring{\Pi}_f := \{ V \in Gr(k,n) \mid f_V = f \} \subset Gr(k,n)$$

$f \in B_{k,n}$

smooth, affine, irreducible, codim $= \ell(f)$

stratifies Grassmannian $Gr(k,n) = \bigsqcup_f \mathring{\Pi}_f$

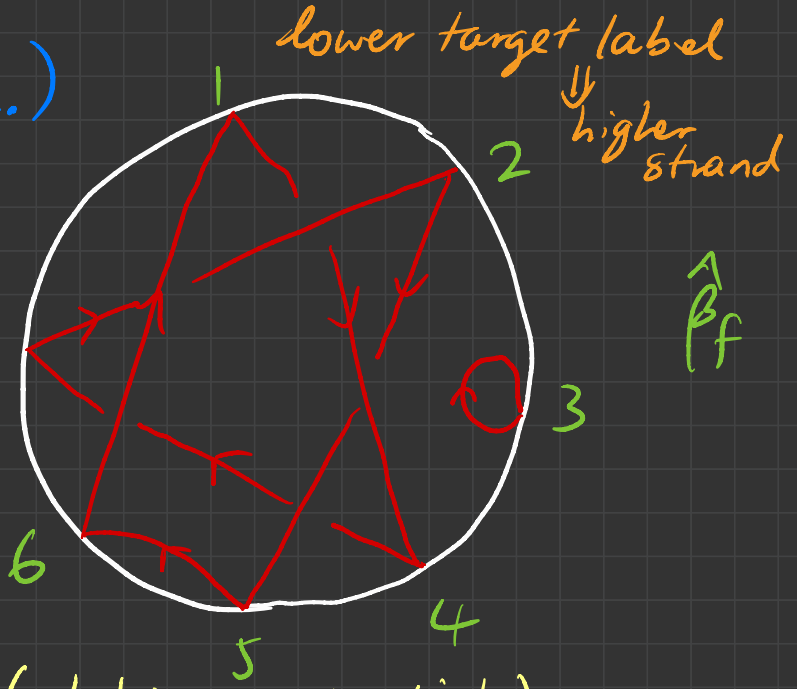
$\Pi_f := \overline{\Pi_f^\circ}$ positroid variety

- normal, Cohen-Macaulay
- compatibly Frobenius split
- appeared in total nonnegativity, Poisson geometry
- $[X_I] = s_{\lambda(I)}$ Schur function $\quad [\Pi_f] = \hat{F}_f$ affine Stanley function

$\in H^*(Gr(k,n))$

Positroid link (Galashin-L.)

1	2	3	4	5	6	7
↓	↓	↓	↓	↓	↓	↓
4	5	3	7	6	8	9



Fomin-Pylyavskyy-Shustin-Thurston (plabic graph link)

Shende-Treumann-Williams-Zaslow (", cyclic rank matrix, ...)

Casals-Gorsky-Gorsky-Simental (Le-diagram)

Galashin-L., in progress (", link on torus)

Theorem [Galashin-L.]

a, q, t

$$H^*_{T,c}(\mathring{\Pi}_f) \cong KR_{top}(\hat{\beta}_f)$$

(with mixed Hodge structure)

triply-graded Khovanov-Rozansky homology

$T \subset PGL(n) \circlearrowright Gr(k,n)$

$R^w_v \subset G/B$ open Richardson variety in G/B

$R^w_v = \Omega^w \cap \Omega_v$

$$H^*_{T,c}(R^w_v) \cong KR^G_{top}(\hat{\beta}_{v,w})$$

Prop

$T \curvearrowright \mathring{\Pi}_f \iff \bar{f} \in S_n \iff \hat{\beta}_f$ is a

free — is a cycle — knot

$\Theta_{k,n} \subset B_{k,n}$ (k,n)-cycles

$\mathcal{X}_f := \mathring{\Pi}_f / T$ positroid configuration space

$\mathcal{X}_{k,n} := \mathring{\Pi}_{k,n} / T = \left\{ \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \middle| \det(v_i, v_{i+1}, \ldots, v_{i+k-1}) \neq 0 \right\} / T$

$\gcd(k,n) = 1$ $\quad f_{k,n}(i) = i+k$

Theorem [Galashin–L.] $\quad \gcd(k,n) = 1$

mixed Hodge poly$(X_{k,n}) = C_{k,n-k}(q,t)$

$\overset{\circ}{\Pi}_{k,n} \subset Gr(k,n)$

$\overset{\circ}{\Pi}_{k,n}/T \quad \dim = k(n-k)-(n-1)$

q,t-rational Catalan number.

$$C_{a,b}(1,1) = C_{a,b} = \frac{1}{(a+b)}\binom{a+b}{a}$$

$$q^{?} C_{a,b}\left(q,\frac{1}{q}\right) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

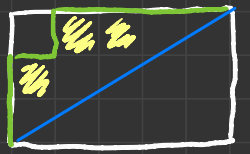
$$C_{a,b}(q,1) = \sum_D q^{\text{area}(D)}$$

$\gcd(q,5) = 1$

$$C_n = C_{n,n+1} = \frac{1}{2n+1}\binom{2n+1}{n}$$

$\#Gr(a,a+b)(\mathbb{F}_q)$

$\longleftrightarrow \#X_{k,n}(\mathbb{F}_q)$



area = 3

- $H^*(X_{k,n})$ has odd cohomology vanishing

When is this true for $H^*(X_f)$ (or $H^*_{T,c}(\tilde{\pi}_f)$)?

$\rightsquigarrow$ algebraic knots

- relation to other geometric models for $C_{a,2}(q,t)$?

$J_{k,n-k}$ $x^k = y^{n-k}$ compactified Jacobians (Gorsky-Mazin, ...)

Hilbert schemes of points in plane (Haiman, Oblomkov-Rozansky)

$$C_{a,b}(q,t) = C_{a,b}(t,q) \quad + \text{ unimodality}$$

$\Uparrow$

curious Lefschetz for $X_{k,n}$.

Haiman, Mellit, Gorsky-Negut, Oblomkov-Rozansky, ...

$\Uparrow$

cluster structure on $\mathring{\pi}_f$ + curious Lefschetz for cluster varieties

Galashin-L. L.-Speyer

$$C_{3,5}(q,t) = q^4 + q^3t + q^2t^2 + qt^3 + t^4 + q^2t + qt^2$$

cf. Hard Lefschetz for $Gr(k,n)$

$H^*(X_{3,8})$ dim $= 3\times 5 - 7 = 8$

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$H^{k,(k,k)}$	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$H^{k,(k-1,k-1)}$					$\mathbb{C}$		$\mathbb{C}$		

$H^{4,(4,4)}$ $H^{4,(3,3)}$

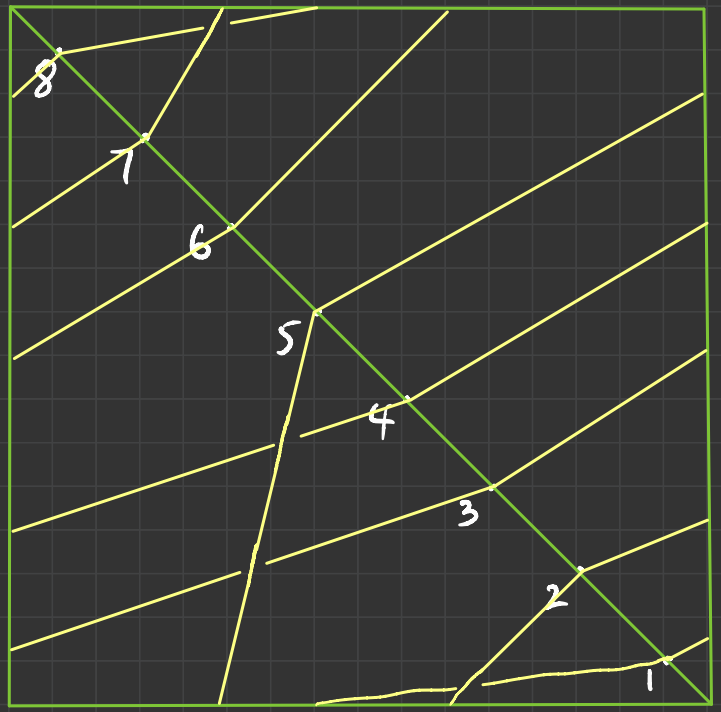
$H^k = \oplus H^{k,(p,q)}$ Deligne splitting

Hodge-Tate type

$H^k = \bigoplus_p H^{k,(p,p)}$

Positroid links on torus $k=3$ $n=8$

$f = [3, 4, 6, 7, 8, 10, 13, 9]$

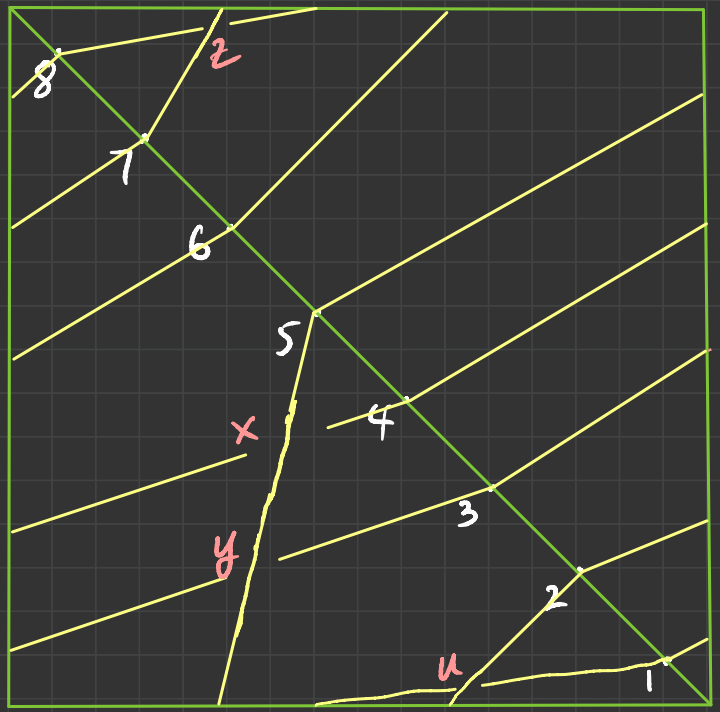


$\tilde{\beta}_f =$

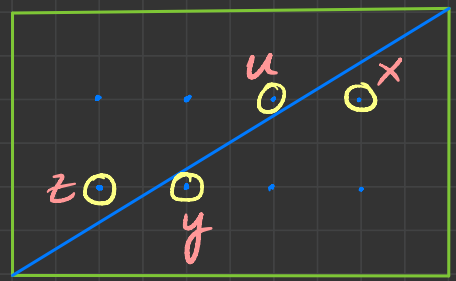
$f_{k,n}(i) = i + k$

$\Downarrow$

$\tilde{\beta}_{f_{k,n}} = \tilde{\beta}_{k,n} = (k, n-k)$-torus link



$\widetilde{\beta}_f$



$\mathcal{T}(f)$

$\mathbb{Z}^2 \cong CH_1(T^2)$

Defn f repetition-free if $\mathcal{T}(f)$ has no repeated elts

Theorem [Galashin-L.] $f \in \Theta_{k,n}$

(1) If f is repetition-free then $T(f)$ is convex ie. $T(f) \cup \{(0,0),(k,n-k)\}$ contains all lattice points in its convex hull and centrally symmetric

(2) Any convex and centrally symmetric subset arises in this way

(3) $\chi(X_f) = \#$ Dyck paths avoiding T_f.

Conjecture f repetition-free

(a) $H^*(X_f)$ has odd-cohomology vanishing

(b) mixed Hodge polynomial (X_f)

$=$

generalized q,t-Catalan numbers

[Gorsky-Hawkes-Schilling-Rainbolt

cf. Negut, Oblomkov-Rozansky]

(c) $=$

specialization of conjecturally Schur-positive series of [Blasiak-Haiman-Morse-Pun-Seelinger]

Knot homologies

Sheaves on Hilbert Scheme

Objects in $D^b_{\text{mixed}}(G/B)$

$\cup$ nice part

even cohomology

algebraic knots

repetition-free positroids

Catalan theory

Schubert / positroid geometry $\longleftrightarrow$? Catalan theory