A spectral gap for the transfer operator on complex projective spaces

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(joint work with Tien-Cuong Dinh)

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Context

- $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, $f$ endomorphism ($k = 1$: rational map)
- for simplicity, no critical periodic points (generic condition)

Goal

Given $\phi: \mathbb{P}^k \to \mathbb{R}$ (or $\mathbb{C}$), understand the Perron-Frobenius (transfer) operator

$$L_\phi(g)(y) = \sum_{f(x) = y} e^{\phi(x)} g(x) \quad \text{for} \quad g: \mathbb{P}^k \to \mathbb{R} \text{ or } \mathbb{C}$$
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\]

More precise goal (A)

Find a Banach space \( (E, \|\cdot\|) \) such that \( \mathcal{L}_\phi: E \to E \)

- has a **spectral gap**
- is analytic in \( \phi \) (\( t \mapsto \mathcal{L}_{\phi + t\psi} \) is analytic in \( t \), as operators \( E \to E \))
Context

- \( \mathbb{P}^k = \mathbb{P}^k(\mathbb{C}), f \) endomorphism \((k = 1: \text{rational map})\)
- for simplicity, no \textit{critical periodic points} (generic condition)

Goal

Given \( \phi: \mathbb{P}^k \to \mathbb{R} \) (or \( \mathbb{C} \)), understand the \textit{Perron-Frobenius (transfer) operator}

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\mathcal{L}_\phi(g)(y) = \sum_{f(x) = y} e^{\phi(x)} g(x) \quad \text{for} \quad g: \mathbb{P}^k \to \mathbb{R} \text{ or } \mathbb{C}
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- has a \textit{spectral gap}
- is analytic in \( \phi \) \((t \mapsto \mathcal{L}_{\phi + t\psi} \text{ is analytic in } t, \text{as operators } E \to E)\)

\[
\lambda^{-n} \mathcal{L}^n(g) \to c_g \nu \text{ exponentially fast } (\sim (r/\lambda)^n)
\]
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More precise goal (A)

Find a Banach space $(E, \|\cdot\|)$ such that $\mathcal{L}_\phi: E \to E$
- has a spectral gap
- is analytic in $\phi$ ($t \mapsto \mathcal{L}_{\phi+t\psi}$ is analytic in $t$, as operators $E \to E$)

$$\mathcal{L}_\phi^n(g)(y) = \sum_{f^n(x) = y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))}g(x)$$
(One) motivation

Problem

Describe orbits of points (in the Julia set)

Deterministic point of view: essentially impossible!

Probabilistic point of view

Given a measure $\nu$, study the sequence of random variables

$$u, u \circ f, u \circ f^2, \ldots$$

for $u: \mathbb{P}^k \to \mathbb{R}$ (observable) of some regularity
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- $\nu$ invariant $\iff U_i := u \circ f^i$ are identically distributed
- The $U_i$’s are not independent, but how close are they to a sequence of independent random variables?

Goal

Prove that $U_i$’s are essentially independent for many natural invariant measures: central limit theorem, deviation theorems...
The equilibrium measure $\mu (\phi = 0; \mathcal{L} = f_*)$

Lyubich, Freire-Lopes-Mañé ’83 for $k = 1$, Fornaess-Sibony ’94, Briend-Duval ’00

$\exists$! measure $\mu$ of maximal entropy, and $\mu$ is such that $f^*\mu = d^k\mu$

Statistical properties for $u$ Hölder continuous

Exponential mixing/decay of correlation, Central Limit Theorem (Dinh-Sibony ’02–’10)
Almost Sure Invariant Principle, law of Iterated Logarithms, ASCLT (Dupont ’10)
Local CLT for $k = 1$, Large Deviation Theorem (Dinh-Nguyen-Sibony ’06, ’10)
...
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...

Essentially ad hoc proofs for the statistical properties

More precise goal (B)

- Obtain these (and other) properties for more general measures, and
- Obtain this by a single approach
Goals

(B)
- Statistical properties for more general measures than $\mu$
- Unified approach
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- Statistical properties for more general measures than $\mu$
- Unified approach

Statistical properties of a random variable $X$ with respect to an invariant measure $\nu$ $\iff$ $t \mapsto \Bbb E(e^{tX})$ with respect to $\nu$, i.e., $t \mapsto \langle e^{tX}, \nu \rangle$
Goals

(B)
- Statistical properties for more general measures than $\mu$
- Unified approach

Statistical properties of a random variable $X$ with respect to an invariant measure $\nu$

$$X = S_n(u) = \sum_{j=1}^{n-1} u \circ f^j$$

$\iff$

$t \mapsto \mathbb{E}(e^{tX})$ with respect to $\nu$, i.e., $t \mapsto \langle e^{tX}, \nu \rangle$

Nagaev, Guivarc'h, Gouëzel, Liverani...

Statistical properties of $X$ are encoded in the coefficients of the Taylor expansion of $t \mapsto \mathbb{E}(e^{tX})$
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Statistical properties of $X$ are encoded in the coefficients of the Taylor expansion of $t \mapsto \mathbb{E}(e^{tX})$

\[ e^{tX} = e^{tu + tu \circ f + \cdots + tu \circ f^{n-1}} \sim \mathcal{L}^n_{0+tu} \]
Goals

(B)
- Statistical properties for more general measures than $\mu$
- Unified approach

Statistical properties of a random variable $X$ with respect to an invariant measure $\nu$

\[ X = S_n(u) = \sum_{j=1}^{n-1} u \circ f^j \]

$\Rightarrow$ \[ t \mapsto \mathbb{E}(e^{tX}) \text{ with respect to } \nu, \text{ i.e., } t \mapsto \langle e^{tX}, \nu \rangle \]

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Statistical properties of $X$ are encoded in the coefficients of the Taylor expansion of $t \mapsto \mathbb{E}(e^{tX})$

\[ e^{tX} = e^{tu + tu \circ f + \cdots + tu \circ f^{n-1}} \sim \mathcal{L}_{0+tu}^n \]
\[ \langle e^{tS_n(u)}, \mu \rangle = \langle \frac{f^* e^{tS_n(u)}}{d^{kn}}, \mu \rangle = \langle \mathcal{L}_{0+tu}^n \frac{1}{d^{kn}}, \mu \rangle \]
(B) Statistical properties for more general measures than $\mu$
- Unified approach

(A) Find a Banach space $(E, \| \cdot \|)$ such that $\mathcal{L}_\phi : E \to E$
- has a spectral gap
- is analytic in $\phi$

Statistical properties of a random variable $X$ with respect to an invariant measure $\nu$

$X = S_n(u) = \sum_{j=1}^{n-1} u \circ f^j$

Nagaev, Guivarc'h, Gouëzel, Liverani...

Statistical properties of $X$ are encoded in the coefficients of the Taylor expansion of $t \mapsto \mathbb{E}(e^{tX})$

$e^{tX} = e^{tu + tu \circ f + \cdots + tu \circ f^{n-1}} \sim \mathcal{L}_{0+tu}^n$

$\langle e^{tS_n(u)}, \mu \rangle = \left\langle \frac{f^n_* e^{tS_n(u)}}{d^{kn}}, \mu \right\rangle = \left\langle \mathcal{L}_{0+tu}^n(1), \mu \right\rangle$
A larger class of invariant measures

\[ \phi = 0 : \quad f^* \mu = d^k \mu \quad \Rightarrow \quad f_\ast \mu = \mu \]
A larger class of invariant measures

\( \phi = 0 : \quad f^* \mu = d^k \mu \quad \Rightarrow \quad f_* \mu = \mu \)

\( \phi : \mathbb{P}^k \rightarrow \mathbb{R} \)

Conformal measure(s)

\( m_\phi \) is a \textit{conformal measure} if it is an eigenvalue for \( \mathcal{L}^* \): \( \exists \lambda \) such that \( \mathcal{L}^* m_\phi = \lambda m_\phi \)

\[
\exists \lambda \in \mathbb{R}, \rho : \mathbb{P}^k \rightarrow \mathbb{R} : \forall g \in \mathcal{C}^0 : \frac{\mathcal{L}^n g(y)}{\lambda^n} \rightarrow c_g \rho \iff \forall \nu : \frac{\mathcal{L}^n \nu}{\lambda^n} \rightarrow m_\phi
\]

Then

- \( m_\phi \) is a conformal measure, \( c_g = \langle m_\phi, g \rangle \), and \( \mathcal{L}(\rho) = \lambda \rho \)
- \( \mu_\phi := \rho m_\phi \) is an invariant measure.
A larger class of invariant measures

\[ \phi = 0 : \quad f^* \mu = d^k \mu \quad \Rightarrow \quad f_* \mu = \mu \]

\[ \phi : \mathbb{P}^k \to \mathbb{R} \]

Conformal measure(s)

\[ m_\phi \text{ is a conformal measure if it is an eigenvalue for } L^* : \exists \lambda \text{ such that } L^* m_\phi = \lambda m_\phi \]

\[ \exists \lambda \in \mathbb{R}, \rho : \mathbb{P}^k \to \mathbb{R} : \forall g \in C^0 : \frac{L^n g (y)}{\lambda^n} \to c_g \rho \iff \forall \nu : \frac{L^n \nu}{\lambda^n} \to m_\phi \]

Then
- \( m_\phi \) is a conformal measure, \( c_g = \langle m_\phi, g \rangle \), and \( L(\rho) = \lambda \rho \)
- \( \mu_\phi := \rho m_\phi \) is an invariant measure. More precisely, an equilibrium state

Equilibrium state(s)

- Pressure \( P(\phi) = \max_\nu \{ h_\nu + \int \phi \nu \} \), where \( h_\nu \) is the metric entropy of the invariant measure \( \nu \).
- \( \mu_\phi \) is an equilibrium state for \( \phi \) if \( P(\phi) = h_{\mu_\phi} + \int \phi \mu_\phi \).
Statistical properties for equilibrium states

\[
\langle e^{tS_n(u)} h, \mu_\phi \rangle = \langle e^{tS_n(u)} h, \rho m_\phi \rangle = \langle \lambda^{-n} \mathcal{L}_\phi^n (\rho e^{tS_n(u)} h), m_\phi \rangle
\]

\[
= \langle \rho \lambda^{-n} \mathcal{L}_{\phi+tu}^n(h), m_\phi \rangle = \langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n(h), \rho m_\phi \rangle = \langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n(h), \mu_\phi \rangle
\]

Statistical properties of a random variable \( u \) with respect to the invariant measure \( \mu_\phi \) (when this exists...)

\[\Leftrightarrow\]

Taylor coefficients of \( t \mapsto \langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n h, \mu_\phi \rangle \) (if they exist...)

\[\Leftrightarrow\]

\( t \mapsto \mathcal{L}_{\phi+tu} \) analytic and has a spectral gap on some \((E, \|\cdot\|)\)
Statistical properties for equilibrium states

\[ \langle e^{tS_n(u)} h, \mu_\phi \rangle = \langle e^{tS_n(u)} h, \rho m_\phi \rangle = \langle \lambda^{-n} L^n_\phi (\rho e^{tS_n(u)} h), m_\phi \rangle = \langle \rho \lambda^{-n} L^n_{\phi+tu}(h), m_\phi \rangle = \langle \lambda^{-n} L^n_{\phi+tu}(h), \rho m_\phi \rangle = \langle \lambda^{-n} L^n_{\phi+tu}(h), \mu_\phi \rangle \]

Statistical properties of a random variable \( u \) with respect to the invariant measure \( \mu_\phi \) (when this exists...)

\[ \Leftrightarrow \]

Taylor coefficients of \( t \mapsto \langle \lambda^{-n} L^n_{\phi+tu} h, \mu_\phi \rangle \) (if they exist...)

\[ \Leftrightarrow \]

\( t \mapsto L_{\phi+tu} \) analytic and has a spectral gap on some \((E, \|\cdot\|)\)

- what is \( \lambda \)? What is the regularity of \( \rho \)?
- How do they depend on \( \phi \)?
- \( \|\lambda^{-n} L^n g - c_g \rho\| \to 0 \)
- \( \lambda^{-1} L \) contraction for \( \|\cdot\| \)??
Theorem 1 (B.-Dinh)

\( \phi: \mathbb{P}^k \to \mathbb{R}, \log^p\text{-continuous for some } p > 2, \Omega(\phi) < \log d. \exists \lambda \in \mathbb{R}, \rho: \mathbb{P}^k \to \mathbb{R} \) such that

\[
\frac{\mathcal{L}_\phi^n g}{\lambda^n} \to c g \rho \quad \forall g: \mathbb{P}^k \to \mathbb{R}
\]

In particular, \( \exists! \) conformal measure \( m_\phi = \lambda^{-1} \mathcal{L}^* m_\phi \), equilibrium state \( \mu_\phi = \rho m_\phi \)

- \( \phi \) Holder: Denker-Urbanski, Przytycki ’90–’91 \( (k = 1) \), Urbanski-Zdunik ’13 \( (k \geq 1) \)
- \( \phi \log^q\text{-continuous } \Leftrightarrow \Omega(\phi, r) \lesssim |\log r|^{-q} \)
Theorem 1 (B.-Dinh)

\[ \phi : \mathbb{P}^k \to \mathbb{R}, \text{log}^p\text{-continuous for some } p > 2, \Omega(\phi) < \log d. \exists \lambda \in \mathbb{R}, \rho : \mathbb{P}^k \to \mathbb{R} \text{ such that} \]

\[ \frac{\mathcal{L}_\phi^n g}{\lambda^n} \to c_g \rho \quad \forall g : \mathbb{P}^k \to \mathbb{R} \]

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- \( \phi \log^q\text{-continuous} \iff \Omega(\phi, r) \lesssim |\log r|^{-q} \)

Classical method

- find \( \lambda \) as an eigenvalue of \( \mathcal{L}^* \) (Schauder-Tikhonov Theorem)
- study the sequence \( \mathcal{L}^n / \lambda^n \) and prove almost periodicity
- converging subsequences \( \Rightarrow \rho \Rightarrow m_\phi, \mu_\phi \)

Here

- We want to find \( \lambda \) intrinsically, as part of our method
- More flexible approach: replace all distortion estimates by a unique, global estimate

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Theorem 2 (B.-Dinh) - New for all \( k \geq 1 \), even for \( \phi \) smooth

For all \( q > 0 \), \( \gamma \leq 2 \) there exist norms \( \| \cdot \|_\infty + \| \cdot \|_{\log q} \leq \| \cdot \|_1 \leq \| \cdot \|_2 \leq \| \cdot \|_{C \gamma} \) depending on \( f \) such that when \( \| \phi \|_2 < \infty \)

1. there exists \( \beta = \beta(\| \phi \|_2) < 1 \) such that:

\[
\| \lambda^{-1} \mathcal{L}_\phi g - \langle m_\phi, g \rangle \rho \|_1 \leq \beta \| g - \langle m_\phi, g \rangle \rho \|_1
\]

2. \( t \mapsto \mathcal{L}_f + t\psi \) is analytic in \( t \)
Theorem 2 (B.-Dinh) - New for all $k \geq 1$, even for $\phi$ smooth

For all $q > 0$, $\gamma \leq 2$ there exist norms $\| \cdot \|_{\infty} + \| \cdot \|_{\log q} \leq \| \cdot \|_{\diamond 1} \approx \| \cdot \|_{\diamond 2} \leq \| \cdot \|_{C^\gamma}$ depending on $f$ such that when $\| \phi \|_{\diamond 2} < \infty$

- there exists $\beta = \beta(\| \phi \|_{\diamond 2}) < 1$ such that:

$$\| \lambda^{-1} \mathcal{L}_\phi g - \langle m_\phi, g \rangle \rho \|_{\diamond 1} \leq \beta \| g - \langle m_\phi, g \rangle \rho \|_{\diamond 1}$$

- $t \mapsto \mathcal{L}_{\phi + t\psi}$ is analytic in $t$

Consequence (A $\Rightarrow$ B)

When $\| \phi \|, \| u \|_{\diamond 2} < \infty$, the sequence $u \circ f^n$ is almost like iid random variables on $(\mathbb{P}^k, \mu_\phi)$: strong ergodic properties (exponential mixing, mixing of all orders, $K$-mixing), Central Limit Theorem, Berry-Esseen Theorem, local Central Limit Theorem, Almost Sure Central Limit Theorem, Large Deviation Theorem and Principle, Almost Sure Invariant Principle, Law of iterated logarithms.

- Related results: Denker-Przytycki-Urbanski, Haydn, Smirnov, Makarov, Ruelle... ($k = 1$); Fornaess-Sibony, Dinh-Nguyen-Sibony, Szostakiewicz-Urbanski-Zdunik... ($k \geq 1$)

- Almost all statistical properties new for $k > 1$, many already for $k = 1$ and/or $\phi = 0$. All new for all $k \geq 1$ for non-Hölder continuous $u$ or $\phi$. 
Theorem 1

\(\phi: \mathbb{P}^k \to \mathbb{R}, \log^p\text{-continuous for some } p > 2, \Omega(\phi) < \log d\).

\(\exists \lambda \in \mathbb{R}, \rho: \mathbb{P}^k \to \mathbb{R}\) such that

\[
\frac{\mathcal{L}_{\phi}^n g}{\lambda^n} \to c g \rho \quad \forall g: \mathbb{P}^k \to \mathbb{R}
\]

In particular, \(\exists!\) conformal measure \(m_\phi = \lambda^{-1} \mathcal{L}^* m_\phi\), equilibrium state \(\mu_\phi = \rho m_\phi\)

Theorem 2

For all \(q > 0, \gamma \leq 2\) there exist norms \(\|\cdot\|_\infty + \|\cdot\|_{\log^q} \leq \|\cdot\|_\diamond \leq \|\cdot\|_2 \leq \|\cdot\|_{C_{\gamma}}\) depending on \(f\) such that when \(\|\phi\|_\diamond < \infty\)

1 there exists \(\beta = \beta(\|\phi\|_\diamond) < 1\) such that:

\[
\|\lambda^{-1} \mathcal{L}_{\phi} g - \langle m_\phi, g \rangle \rho\|_\diamond \leq \beta \|g - \langle m_\phi, g \rangle \rho\|_\diamond
\]

2 \(t \mapsto \mathcal{L}_{\phi+t\psi}\) is analytic in \(t\)
Hölder and $\log^q$-continuous functions

$$\phi \in C^\gamma \iff \Omega(\phi, r) \lesssim r^\gamma$$
$$\phi \in \log^q \iff \Omega(\phi, r) \lesssim |\log r|^{-q}$$

Viewpoint from interpolation theory:

$$\phi = \phi_1^1 + \phi_2^2, \quad \|\phi_2^2\|_\infty < \epsilon, \quad \|\phi_1^1\|_{C^2} < ??$$

$$\phi \in C^\gamma \iff \|\phi_1^1\|_{C^2} \lesssim (1/\epsilon)^{2/\gamma}$$
$$\phi \in \log^q \iff \|\phi_1^1\|_{C^2} \lesssim e^{(1/\epsilon)^{1/q}}$$

We will need summable errors $\Rightarrow \epsilon = 1/j^2$

$\Rightarrow q > 2$: $\phi$ can be approximated with functions $\phi_j := \phi_1^{1/j}$ whose $C^2$ norms diverge sub-exponentially in $j$
Idea of the method

Classical

- find $\lambda$ as an eigenvalue of $L^*$ (Schauder-Tikhonov Theorem)
- study the sequence $L^n/\lambda^n$ and prove \textit{almost periodicity}
- converging subsequences $\Rightarrow \rho \Rightarrow m_\phi, \mu_\phi$

Here

- We want to find $\lambda$ \textit{intrinsically}, as part of our method
- We just normalize (for $g$ positive) by $\int L^n g \text{ Leb}$, or $\min L^n g$, and we will see the exponential behaviour later.

For simplicity: $\phi \in C^2, g = 1$. Denote $1_n := L^n 1$ and $1_n^* = 1_n / \min 1_n$.

Idea

We prove that $\max 1_n^* = \max 1_n / \min 1_n$ is bounded
Method: finding $\lambda$

Idea

We prove that $\max 1^*_n = \max 1_n / \min 1_n$ is bounded

Then:

$$
\begin{align*}
\max 1_{n+m} &\leq \max 1_n \cdot \max 1_m \\
\min 1_{n+m} &\geq \min 1_n \cdot \min 1_m \\
\max 1_n / \min 1_n &\leq C
\end{align*}
$$

$\Rightarrow$

$\lambda := \inf_n (\max 1_n)^{1/n} := \sup_n (\min 1_n)^{1/n}$

To bound $\max / \min$, we bound $\Omega / \min = (\max - \min) / \min$

We need to bound $\Omega(1^*_n)$
Bounding the oscillation ($k = 1$ for simplicity)

Bound on $dd^c \Rightarrow$ bound on oscillation

Lemma (Heuristic version)

$dd^c g \leq dd^c h$ then $\Omega(g, r) \lesssim \Omega(h, r)$.

if $dd^c g \leq R$ with continuous potentials, then the family $g_n$ is equicontinuous.

Here we want $dd^c 1^n \leq R$ for some uniform $R$, for which we control the regularity of the potential.
Bounding the oscillation ($k = 1$ for simplicity)

Bound on $dd^c \Rightarrow$ bound on oscillation

Lemma (Heuristic version)

$dd^c g \leq dd^c h$ then $\Omega(g, r) \lesssim \Omega(h, r)$.

Lemma (More precise version)

- $\Omega(g, r) \lesssim \Omega(h, \sqrt{r}) + A\sqrt{r}$.
- If $dd^c g_n \leq R$ with continuous potentials, then the family $g_n$ is equicontinuous.

Here we want

$dd^c 1_n^* \leq R$

for some uniform $R$, for which we control the regularity of the potential.
Bounding the oscillation of $\mathbb{1}_n^*$

Development of $\mathbb{1}_n$

$$dd^c \mathbb{1}_n = dd^c \left( \sum_{f^n(x) = y} e^\phi + \phi(f(x)) + \ldots + \phi(f^{n-1}(x)) \mathbb{1} \right)$$
Bounding the oscillation of $1_n^*$

Development of $1_n$

$$dd^c 1_n = dd^c \left( \sum_{f^n(x) = y} e^{\Phi + \Phi(f(x)) + \ldots + \Phi(f^{n-1}(x))} \right)$$

$$= \sum_{f^n(x) = y} e^{\Phi + \Phi(f(x)) + \ldots + \Phi(f^{n-1}(x))} \left( \sum_{j=0}^{n-1} dd^c \phi(f^j(x)) + \sum_{j,l=0}^{n-1} \partial \phi(f^j(x)) \wedge \bar{\partial} \phi(f^l(x)) \right)$$
Bounding the oscillation of $1_n^*$

Development of $1_n$

\[
dd^c 1_n = dd^c \left( \sum_{f^n(x) = y} e^{\Phi + \Phi(f(x)) + \ldots + \Phi(f^{n-1}(x))} 1 \right)
\]

\[
= \sum_{f^n(x) = y} e^{\Phi + \Phi(f(x)) + \ldots + \Phi(f^{n-1}(x))} \left( \sum_{j=0}^{n-1} dd^c \phi(f^j(x)) + \sum_{j,l=0}^{n-1} \partial \phi(f^j(x)) \land \overline{\partial} \phi(f^l(x)) \right)
\]

\[\vdots\]

(more complicated with $g, \phi$ less regular)

\[
dd^c 1_n^* \lesssim \sum_{j=0}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \Omega(1_{n-j}) \| \phi \|_{C^2} f^{j-1}_* \text{Leb}
\]

\[
\lesssim \sum_{j=0}^{\infty} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j f^{j-1}_* \text{Leb}
\]

Ok for mass. We still need to estimate the oscillation of the potential of the RHS. But what is this potential?
(Dynamical) potentials

$$\text{Leb} = \mu + dd^c u_0 \quad f_*^j \text{Leb} = \mu + dd^c u_j$$

- $u_0$ is the Green function, which is $\gamma$-Hölder.
- Up to a Hölder continuous function, the potential of

$$\sum_{j=0}^{\infty} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j f_*^{j-1} \text{Leb} \quad \text{is given by} \quad \sum_{j=0}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j u_j$$
(Dynamical) potentials

\[ \text{Leb} = \mu + dd^c u_0 \quad f^j \text{Leb} = \mu + dd^c u_j \]

- \( u_0 \) is the Green function, which is \( \gamma \)-Hölder.
- Up to a Hölder continuous function, the potential of

\[
\sum_{j=0}^{\infty} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j f^{j-1} \text{Leb} \quad \text{is given by} \quad \sum_{j=0}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j u_j
\]

**Lemma**

1. \( u_j \) is \( \gamma/2^j \) Hölder.
2. \( \|u_j\|_\infty \lesssim d^n/\delta^n \) for all \( \delta < d \).

\[
\Rightarrow \sum_{j=0}^{\infty} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j u_j \in \log^p \quad \forall p
\]

\[
\Rightarrow \|1_n^*\|_{\log^p} < C_p \quad \forall n, p
\]
When $\phi$ is not $C^2$

$$\phi \in \log^q \Rightarrow \begin{cases} 
\phi = \phi^1_j + \phi^2_j \\
\|\phi^2_j\|_\infty \leq 1/j^2 \\
\|\phi^1_j\|_{C^2} \leq e^{j^2/q} \end{cases} \text{← sub-exponential}$$

$$dd^c 1^*_n \lesssim \sum_{j=1}^n \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \Omega(1^*_n) \|\phi^1_j\|_{C^2} f^{j-1} Leb$$

$\Rightarrow$ Existence and uniqueness of equilibrium state and conformal measure for all $\phi$ with $\|\phi\|_{\log^q} < \infty$ for some $q > 2$ (Theorem 1)
When $\phi$ is not $C^2$

$$\phi \in \log^q \Rightarrow \begin{cases} \phi = \phi_j^1 + \phi_j^2 \\ \|\phi_j^2\|_\infty \leq 1/j^2 \\ \|\phi_j^1\|_{C^2} \leq e^{j^2/q} \leq \text{sub-exponential} \end{cases}$$

$$dd^c 1_n^* \lesssim \sum_{j=1}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \Omega(1_{n-j}) \|\phi_j^1\|_{C^2} f_j^{*-1} \text{Leb}$$

$\Rightarrow$ Existence and uniqueness of equilibrium state and conformal measure for all $\phi$ with $\|\phi\|_{\log^q} < \infty$ for some $q > 2$ (Theorem 1)

Second (and main) goal: find a norm so that this convergence becomes a contraction in a suitable space of functions.
Norm and spectral gap

First consider the case $\phi = 0$

**DSH norm (Dinh-Sibony)**

$$\|g\|_{DSH} = \min \|R^+\|$$, where $dd^c g = R^+ - R^-$, $R^\pm$ positive measures

Then

$$\left\| \frac{f_* g}{d} \right\|_{DSH} \leq \frac{1}{d} \left\| f_* R^+ - f_* R^- \right\| = \frac{1}{d} \|g\|_{DSH}$$
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**DSH norm (Dinh-Sibony)**
\[ \|g\|_{DSH} = \min \|R^+\|, \text{ where } dd^c g = R^+ - R^-, \ R^\pm \text{ positive measures} \]

Then
\[ \left\| \frac{f^* g}{d} \right\|_{DSH} \leq \frac{1}{d} \|f^* R^+ - f^* R^-\| = \frac{1}{d} \|g\|_{DSH} \]

Here, if we try to do the same
\[
 dd^c \mathcal{L}_\phi(g) \sim \sum_{f(x) = y} e^{\phi(x)} \, dd^c g + g dd^c \phi \, e^\phi + e^\phi \partial g \overline{\partial} \phi + e^\phi \partial^2 \phi \overline{\partial} g
\]
\[
 dd^c \mathcal{L}^n_\phi(g) \sim \ldots
\]
\[ dd^c \mathcal{L}_\phi(g) \sim \sum_{f(x) = y} e^{\Phi(x)} dd^c g + \overline{d} \Phi \overline{\partial} \bar{\Phi} + \overline{\Phi} \partial \bar{\Phi} \overline{\partial} g \]

- The operator \( dd^c \) is complex (commutes with \( f^* \)). Here non complex perturbation \( (f^*(e^{\Phi} \cdot )) \). No way to keep complex norm like \( DSH \) even if \( \phi \) is smooth.
- Serious problem for norm is given by mixed terms.
- \( f^* \) does not work well with Hölder, so need weaker.

Idea 0

Use a norm with bound on \( dd^c \) + regularity; then obtain spectral gap on "real" norm by interpolation. We use pluripotential theory to study the norm with \( dd^c \).

Idea 1

Use something like \( \| \cdot \|_{p} \approx \| \cdot \|_{DSH} + \| \cdot \|_{\log p} \approx \| \nu \|_{p} + \| u \nu \|_{\log p} \) for measures.)
The operator $dd^c$ is complex (commutes with $f_*$). Here non complex perturbation $(f_*(e^\Phi \cdot))$. No way to keep complex norm like $DSH$ even if $\phi$ is smooth.

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**Idea 0**

Use a norm with bound on $dd^c +$ regularity; then obtain spectral gap on "real" norm by interpolation. We use pluripotential theory to study the norm with $dd^c$. 
\[ dd^c L_\phi(g) \sim \sum_{f(x)=y} e^{\Phi(x)} dd^c g + g dd^c \phi e^\phi + e^\phi \partial g \overline{\partial} \phi + e^\phi \partial \phi \overline{\partial} g \]

- The operator \( dd^c \) is complex (commutes with \( f^* \)). Here non complex perturbation \( (f^*(e^\phi \cdot)) \). No way to keep complex norm like \( DSH \) even if \( \phi \) is smooth.
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Use a norm with bound on \( dd^c \) + regularity; then obtain spectral gap on "real" norm by interpolation. We use pluripotential theory to study the norm with \( dd^c \).

**Idea 1**

Use something like \( \| \cdot \|_{p} \cong \| \cdot \|_{DSH} + \| \cdot \|_{\log^p} (\| \nu \|_{p} \cong \| \nu \|_{*} + \| u_{\nu} \|_{\log^p} \) for measures)
\[ \| \cdot \|_p \cong \| \cdot \|_{DSH} + \| \cdot \|_{\log^p} \]

**Lemma**

\[ \| \partial g \wedge \overline{\partial} h \|_p \leq \| g \|_p \| h \|_p \]
\[ \| \cdot \|_p \trianglerighteq \| \cdot \|_{DSH} + \| \cdot \|_{\log p} \]

**Lemma**

\[ \| \partial g \wedge \overline{\partial} h \|_p \leq \| g \|_p \| h \|_p \]

... but other problems: loss of regularity!

\[
\frac{dd^c 1_n}{\lambda^n} \lesssim \left( \frac{e^{\Omega(\phi)}}{d} \right)^n f_*^n dd^c g + \sum_{j=1}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \| \mathcal{L}^{n-j} g \|_\infty f_{j-1}^{-1} dd^c \phi + \ldots
\]

The potential of the RHS is

\[
\left( \frac{e^{\Omega(\phi)}}{d} \right)^n f_*^n g + \sum_{j=1}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \| \mathcal{L}^{n-j} g \|_\infty f_{j-1}^{-1} \phi + \ldots \in ???
\]
\[ \| \cdot \|_p \cong \| \cdot \|_{DSH} + \| \cdot \|_{\log p} \]

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... but other problems: loss of regularity!

\[ \frac{dd^c 1_n}{\lambda^n} \lesssim \left( \frac{e^{\Omega(\phi)}}{d} \right)^n f^*_n dd^c g + \sum_{j=1}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \left\| \mathcal{L}^{n-j} \frac{g}{\lambda^{n-j}} \right\|_{\infty} f^{j-1} dd^c \phi + \ldots \]

The potential of the RHS is
\[ \left( \frac{e^{\Omega(\phi)}}{d} \right)^n f^*_n g + \sum_{j=1}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \left\| \mathcal{L}^{n-j} \frac{g}{\lambda^{n-j}} \right\|_{\infty} f^{j-1} \phi + \ldots \in ??? \]

**Lemma**
\[ \| d^{-j} f^*_j \phi \|_{\log p} \leq c_p(A) A^n \| \phi \|_{\log p} \]
for all \( A > 1 \)

**Theorem**
\[ \| \phi \|_p < \infty \text{ then } \| f^*_j \phi / d^j \|_{\infty} \to 0 \]
exponentially (precise bounds).

\[ \Rightarrow \left( \frac{e^{\Omega(\phi)}}{d} \right)^n f^*_n g + \sum_{j=1}^{n} \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \left\| \mathcal{L}^{n-j} \frac{g}{\lambda^{n-j}} \right\|_{\infty} f^{j} \phi \in \log^q \text{ for some explicit } q < p \]

\[ \Rightarrow \text{convergence } \| \lambda^{-n} \mathcal{L}_\phi^n g \|_q \to 0, \text{ uniform in } g, \text{ but no spectral gap yet!} \]
Idea 2: a dynamical norm

**Definition**

\[ \| R \|_{\alpha, p} := \min c : R \leq c \sum_j \alpha^j f_j^* S \text{ for some } \| S \|_p \leq 1. \]

By definition: \( \| f_\ast R \|_{\alpha, p} \leq \frac{1}{\alpha} \| R \|_{\alpha, p} \)

\[
\Rightarrow \left\| \left( \frac{e^{\Omega(\Phi)}}{d} \right)^n f_\ast^n dd^c g + \sum_{j=1}^n \left( \frac{e^{\Omega(\Phi)}}{d} \right)^j \left\| \frac{L^{n-j}g}{\lambda^{n-j}} \right\|_\infty f_j^{-1} dd^c \phi \right\|_{\alpha, p} \\
\leq \left( \frac{e^{\Omega(\Phi)}}{\alpha d} \right)^n \| dd^c g \|_{\alpha, p} + \sum_{j=1}^n \left( \frac{e^{\Omega(\Phi)}}{\alpha d} \right)^j \left\| \frac{L^{n-j}g}{\lambda^{n-j}} \right\|_\infty \| dd^c \phi \|_{\alpha, p} \\
\leq c_n \| dd^c g \|_{\alpha, p} \rightarrow 0!
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By definition: \[ \| f_* R \|_{\alpha,p} \leq \frac{1}{\alpha} \| R \|_{\alpha,p} \]

\[
\Rightarrow \left\| \left( \frac{e^{\Omega(\phi)}}{d} \right)^n f_* dd^c g + \sum_{j=1}^n \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \left\| \frac{L^{n-j} g}{\lambda^{n-j}} \right\|_{\infty} f_j^{-1} dd^c \phi \right\|_{\alpha,p} \\
\leq \left( \frac{e^{\Omega(\phi)}}{\alpha d} \right)^n \| dd^c g \|_{\alpha,p} + \sum_{j=1}^n \left( \frac{e^{\Omega(\phi)}}{\alpha d} \right)^j \left\| \frac{L^{n-j} g}{\lambda^{n-j}} \right\|_{\infty} \| dd^c \phi \|_{\alpha,p} \\
\leq c_n \| dd^c g \|_{\alpha,p} \rightarrow 0!
\]

Where is the problem?
Idea 2: a dynamical norm

**Definition**

\[ \|R\|_{\alpha,p} := \min c : R \leq c \sum_{j} \alpha^j f_j^* S \text{ for some } \|S\|_p \leq 1. \]

By definition: \( \|f_* R\|_{\alpha,p} \leq \frac{1}{\alpha} \|R\|_{\alpha,p} \)

\[
\Rightarrow \left\| \left( \frac{e^{\Omega(\phi)}}{d} \right)^n f_*^{dd^c} g + \sum_{j=1}^n \left( \frac{e^{\Omega(\phi)}}{d} \right)^j \| \frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}} \|_\infty f_j^{-1}^{dd^c} \phi \right\|_{\alpha,p}
\]

\[
\leq \left( \frac{e^{\Omega(\phi)}}{\alpha d} \right)^n \| dd^c g \|_{\alpha,p} + \sum_{j=1}^n \left( \frac{e^{\Omega(\phi)}}{\alpha d} \right)^j \| \frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}} \|_\infty \| dd^c \phi \|_{\alpha,p}
\]

\[
\leq c_n \| dd^c g \|_{\alpha,p} \rightarrow 0!
\]

Where is the problem? We did not consider the mixed terms!
The problem of the mixed terms

**Definition**

\[ \| g \|_{\alpha,p} := \| d d^c g \| := \min c : d d^c g \leq c \sum_j \alpha^j f^j S \text{ for some } \| S \|_p \leq 1. \]

Then we have

\[ d d^c (g h) = g d d^c h + h d d^c g + i \partial g \wedge \bar{\partial} h + i \partial h \wedge \bar{\partial} g \]

\[ \| d d^c g h \|_{\alpha,p} \leq \| g \|_\infty \| d d^c h \|_{\alpha,p} + \| h \|_\infty \| d d^c g \|_{\alpha,p} + \| i \partial g \wedge \bar{\partial} h + i \partial h \wedge \bar{\partial} g \|_{\alpha,p} \]
The problem of the mixed terms

**Definition**

\[ \|g\|_{\alpha,p} := \|dd^c g\| := \min c : dd^c g \leq c \sum_j \alpha^j f^j S \text{ for some } \|S\|_p \leq 1. \]

Then we have

\[ dd^c(gh) = gdd^c h + hdd^c g + i\partial g \wedge \overline{\partial} h + i\partial h \wedge \overline{\partial} g \]

\[ \|dd^c gh\|_{\alpha,p} \leq \|g\|_{\infty} \|dd^c h\|_{\alpha,p} + \|h\|_{\infty} \|dd^c g\|_{\alpha,p} + \|i\partial g \wedge \overline{\partial} h + i\partial h \wedge \overline{\partial} g\|_{\alpha,p} \]

**Modified definition**

\[ \|g\|^2_{\alpha,p} := \|i\partial g \wedge \overline{\partial} g\| := \min c : i\partial g \wedge \overline{\partial} g \leq c \sum_j \alpha^j f^j S, \|S\|_p \leq 1. \]
\[ \|g\|_{\alpha,p}^2 := \|i\partial g \wedge \bar{\partial} g\| := \min c: i\partial g \wedge \bar{\partial} g \leq c \sum_j \alpha^j f^j S \text{ for some } \|S\|_p \leq 1. \]

- ok for mixed terms
- still good shifting property (less direct)
- spectral gap
\[ \|g\|_{\alpha, p}^2 := \|i\partial g \wedge \bar{\partial}g\| := \min c: i\partial g \wedge \bar{\partial}g \leq c \sum_{j} \alpha^j f^j S \text{ for some } \|S\|_p \leq 1. \]

- ok for mixed terms
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- spectral gap?

Main issue... all the method was based on the

**Lemma**

\[ dd^c g \leq dd^c h \Rightarrow \Omega(g, r) \lesssim \Omega(h, r) \]

We need

**Lemma**

\[ \partial g \wedge \bar{\partial}g \leq dd^c h \Rightarrow \Omega(g, r) \lesssim \Omega(h, r) \]
\[ \|g\|_{\alpha,p}^2 := \|i \partial g \wedge \bar{\partial} g\| := \min c: i \partial g \wedge \bar{\partial} g \leq c \sum_j \alpha^j f^*_j S \text{ for some } \|S\|_p \leq 1. \]

- ok for mixed terms
- still good shifting property (less direct)
- spectral gap? YES!

Main issue... all the method was based on the

**Lemma**

\[ dd^c g \leq dd^c h \Rightarrow \Omega(g, r) \lesssim \Omega(h, r) \]

We need

**Lemma**

\[ \partial g \wedge \bar{\partial} g \leq dd^c h \Rightarrow \Omega(g, r) \lesssim \Omega(h, r) \]

Much more involved but true!
Spectral gap(s) by interpolation

We have a spectral gap for the norm $\|g\|_{\alpha,p}^2 := \| \partial g \wedge \overline{\partial g} \|_{\alpha,p}$.

⇒ We build a $\gamma$-Hölder-like norm from $\| \cdot \|_{\alpha,p}$:

**Definition**

$$\|g\|_{\alpha,p,\gamma} := \min c : \forall 0 < \epsilon < 1 : \begin{cases} g = g_\epsilon^1 + g_\epsilon^2 \\ \|g_\epsilon^2\|_\infty \leq c\epsilon \\ \|g_\epsilon^1\|_{\alpha,p} \leq c(1/\epsilon)^{1/\gamma} \end{cases}$$

- $\log^q \leq \| \cdot \|_{\alpha,p,\gamma} \leq \| \cdot \|_{\epsilon\gamma}$
- Interpolation techniques (all the method is stable under "sub-exponential perturbations"):

Spectral gap for $\| \cdot \|_{\alpha,p,\gamma} \lesssim C^\gamma$