A spectral gap for the transfer operator on complex projective spaces

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(joint work with Tien-Cuong Dinh)

CNRS and Université de Lille

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Context

- $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, f endomorphism ($k = 1$: rational map)
- for simplicity, no *critical periodic points* (generic condition)

Goal

Given $\phi\colon \mathbb{P}^k \to \mathbb{R}$ (or $\mathbb{C}$), understand the *Perron-Frobenius (transfer) operator*

$$\mathcal{L}_\phi(g)(y) = \sum_{f(x)=y} e^{\phi(x)} g(x) \quad \text{for} \quad g\colon \mathbb{P}^k \to \mathbb{R} \text{ or } \mathbb{C}$$

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More precise goal (A)

Find a Banach space $(E, \|\cdot\|)$ such that $\mathcal{L}_\phi\colon E \to E$

- has a *spectral gap*
- is analytic in ϕ ($t \mapsto \mathcal{L}_{\phi+t\psi}$ is analytic in t, as operators $E \to E$)

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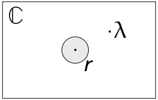

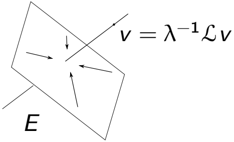

$\lambda^{-n}\mathcal{L}^n(g) \to c_g v$ exponentially fast ($\sim (r/\lambda)^n$)

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$$\mathcal{L}^n_\phi(g)(y) = \sum_{f^n(x)=y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))} g(x)$$

(One) motivation

Problem

Describe orbits of points (in the Julia set)

Deterministic point of view: essentially impossible!

Probabilistic point of view

Given a measure ν, study the sequence of *random variables*

$$u, u \circ f, u \circ f^2, \ldots$$

for $u \colon \mathbb{P}^k \to \mathbb{R}$ (observable) of some regularity

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- ν invariant $\Leftrightarrow U_i := u \circ f^i$ are identically distributed
- The U_i's are not independent, but how close are they to a sequence of independent random variables?

Goal

Prove that U_i's are *essentially independent* for many natural invariant measures: central limit theorem, deviation theorems...

The equilibrium measure μ $(\phi = 0; \mathcal{L} = f_*)$

Lyubich, Freire-Lopes-Mañé '83 for $k = 1$, Fornaess-Sibony '94, Briend-Duval '00

$\exists!$ measure μ of maximal *entropy*, and μ is such that $f^*\mu = d^k\mu$

Statistical properties for u Hölder continuous

Exponential mixing/decay of correlation, Central Limit Theorem (Dinh-Sibony '02-'10)
Almost Sure Invariant Principle, law of Iterated Logarithms, ASCLT (Dupont '10)
Local CLT for $k = 1$, Large Deviation Theorem (Dinh-Nguyen-Sibony '06, '10)
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Essentially *ad hoc* proofs for the statistical properties

More precise goal (B)

- Obtain these (and other) properties for more general measures, and
- Obtain this by a single approach

Goals

(B)

- Statistical properties for more general measures than μ
- Unified approach

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Statistical properties of a random variable X with respect to an invariant measure ν $\Leftrightarrow$ $t \mapsto \mathbb{E}(e^{tX})$ with respect to ν, i.e., $t \mapsto \langle e^{tX}, \nu \rangle$

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$$X = S_n(u) = \sum_{j=1}^{n-1} u \circ f^j$$

Nagaev, Guivarc'h, Gouëzel, Liverani...

Statistical properties of X are encoded in the coefficients of the Taylor expansion of $t \mapsto \mathbb{E}(e^{tX})$

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Find a Banach space $(E, \|\cdot\|)$ such that $\mathcal{L}_\phi : E \to E$

- has a *spectral gap*
- is analytic in ϕ

Statistical properties of a random variable X with respect to an invariant measure ν

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A larger class of invariant measures

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$\phi \colon \mathbb{P}^k \to \mathbb{R}$

Conformal measure(s)

m_ϕ is a *conformal measure* if it is an eigenvalue for $\mathcal{L}^*$: $\exists \lambda$ such that $\mathcal{L}^* m_\phi = \lambda m_\phi$

$$\exists \lambda \in \mathbb{R}, \rho \colon \mathbb{P}^k \to \mathbb{R} : \forall g \in \mathcal{C}^0 \colon \frac{\mathcal{L}^n g(y)}{\lambda^n} \to c_g \rho \Leftrightarrow \forall \nu \colon \quad \frac{\mathcal{L}^{*n}\nu}{\lambda^n} \to m_\phi$$

Then

- m_ϕ is a conformal measure, $c_g = \langle m_\phi, g \rangle$, and $\mathcal{L}(\rho) = \lambda\rho$
- $\mu_\phi := \rho m_\phi$ is an invariant measure.

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Then

- m_ϕ is a conformal measure, $c_g = \langle m_\phi, g \rangle$, and $\mathcal{L}(\rho) = \lambda\rho$
- $\mu_\phi := \rho m_\phi$ is an invariant measure. More precisely, an *equilibrium state*

Equilibrium state(s)

- Pressure $P(\phi) = \max_\nu \left\{ h_\nu + \int \phi\nu \right\}$, where h_ν is the metric entropy of the invariant measure ν.
- μ_ϕ is an *equilibrium state for* ϕ if $P(\phi) = h_{\mu_\phi} + \int \phi \mu_\phi$.

Statistical properties for equilibrium states

$$\boxed{\left\langle e^{tS_n(u)} h, \mu_\phi \right\rangle} = \left\langle e^{tS_n(u)} h, \rho m_\phi \right\rangle = \left\langle \lambda^{-n} \mathcal{L}_\phi^n (\rho e^{tS_n(u)} h), m_\phi \right\rangle$$

$$= \left\langle \rho \lambda^{-n} \mathcal{L}_{\phi+tu}^n (h), m_\phi \right\rangle = \left\langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n (h), \rho m_\phi \right\rangle = \boxed{\left\langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n (h), \mu_\phi \right\rangle}$$

Statistical properties of a random variable u with respect to the invariant measure μ_ϕ (when this exists...)

$\Leftrightarrow$

Taylor coefficients of $t \mapsto \left\langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n h, \mu_\phi \right\rangle$ (if they exist...)

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$t \mapsto \mathcal{L}_{\phi+tu}$ analytic and has a spectral gap on some $(E, \|\cdot\|)$

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Statistical properties of a random variable u with respect to the invariant measure μ_ϕ (when this exists...) $\Leftrightarrow$ Taylor coefficients of $t \mapsto \left\langle \lambda^{-n} \mathcal{L}_{\phi+tu}^n h, \mu_\phi \right\rangle$ (if they exist...) $\Leftarrow$ $t \mapsto \mathcal{L}_{\phi+tu}$ analytic and has a spectral gap on some $(E, \|\cdot\|)$

- what is λ? What is the regularity of ρ?
- How do they depend on ϕ?
- $\|\lambda^{-n}\mathcal{L}^n g - c_g \rho\|_? \to 0$
- $\lambda^{-1}\mathcal{L}$ contraction for $\|\cdot\|_{??}$

Theorem 1 (B.-Dinh)

$\phi\colon \mathbb{P}^k \to \mathbb{R}$, $\log^p$-continuous for some $p > 2$, $\Omega(\phi) < \log d$. $\exists \lambda \in \mathbb{R}$, $\rho\colon \mathbb{P}^k \to \mathbb{R}$ such that

$$\frac{\mathcal{L}_\phi^n g}{\lambda^n} \to c_g \rho \qquad \forall g\colon \mathbb{P}^k \to \mathbb{R}$$

In particular, $\exists!$ conformal measure $m_\phi = \lambda^{-1}\mathcal{L}^* m_\phi$, equilibrium state $\mu_\phi = \rho m_\phi$

- ϕ Holder: Denker-Urbanski, Przytycki '90-'91 ($k = 1$), Urbanski-Zdunik '13 ($k \geqslant 1$)
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Classical method

- find λ as an eigenvalue of $\mathcal{L}^*$ (Schauder-Tikhonov Theorem)
- study the sequence $\mathcal{L}^n/\lambda^n$ and prove *almost periodicity*
- converging subsequences $\Rightarrow \rho \Rightarrow m_\phi, \mu_\phi$

Here

- We want to find λ *intrinsically*, as part of our method
- More flexible approach: replace all distortion estimates by a *unique, global estimate*

Theorem 2 (B.-Dinh) - New for all $k \geqslant 1$, even for ϕ smooth

For all $q > 0, \gamma \leqslant 2$ there exist norms $\|\cdot\|_\infty + \|\cdot\|_{\log^q} \leqslant \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \leqslant \|\cdot\|_{\mathcal{C}^\gamma}$ depending on f such that when $\|\phi\|_{\diamond_2} < \infty$

1. there exists $\beta = \beta(\|\phi\|_{\diamond_2}) < 1$ such that:

$$\left\|\lambda^{-1}\mathcal{L}_\phi g - \langle m_\phi, g\rangle \rho\right\|_{\diamond_1} \leqslant \beta \left\|g - \langle m_\phi, g\rangle \rho\right\|_{\diamond_1}$$

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Consequence (A $\Rightarrow$ B)

When $\|\phi\|, \|u\|_{\diamond_2} < \infty$, the sequence $u \circ f^n$ is *almost like* iid random variables on $(\mathbb{P}^k, \mu_\phi)$: strong ergodic properties (exponential mixing, mixing of all orders, Kmixing), Central Limit Theorem, Berry-Esseen Theorem, local Central Limit Theorem, Almost Sure Central Limit Theorem, Large Deviation Theorem and Principle, Almost Sure Invariant Principle, Law of iterated logarithms.

- Related results: Denker-Przytycki-Urbanski, Haydn, Smirnov, Makarov, Ruelle... ($k = 1$); Fornaess-Sibony, Dinh-Nguyen-Sibony, Szostakiewicz-Urbanski-Zdunik... ($k \geqslant 1$)
- Almost all statistical properties new for $k > 1$, many already for $k = 1$ and/or $\phi = 0$. All new for all $k \geqslant 1$ for non-Hölder continuous u or ϕ.

Theorem 1

$\phi\colon \mathbb{P}^k \to \mathbb{R}$, $\log^p$-continuous for some $p > 2$, $\Omega(\phi) < \log d$. $\exists \lambda \in \mathbb{R}$, $\rho\colon \mathbb{P}^k \to \mathbb{R}$ such that

$$\frac{\mathcal{L}_\phi^n g}{\lambda^n} \to c_g \rho \qquad \forall g\colon \mathbb{P}^k \to \mathbb{R}$$

In particular, $\exists!$ conformal measure $m_\phi = \lambda^{-1}\mathcal{L}^* m_\phi$, equilibrium state $\mu_\phi = \rho m_\phi$

Theorem 2

For all $q > 0, \gamma \leqslant 2$ there exist norms $\|\cdot\|_\infty + \|\cdot\|_{\log^q} \leqslant \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \leqslant \|\cdot\|_{\mathcal{C}^\gamma}$ depending on f such that when $\|\phi\|_{\diamond_2} < \infty$

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Hölder and $\log^q$-continuous functions

$$\phi \in \mathcal{C}^\gamma \Leftrightarrow \quad \Omega(\phi, r) \lesssim r^\gamma$$

$$\phi \in \log^q \Leftrightarrow \quad \Omega(\phi, r) \lesssim |\log r|^{-q}$$

Viewpoint from interpolation theory:

$$\phi = \phi^1_\epsilon + \phi^2_\epsilon, \quad \left\|\phi^2_\epsilon\right\|_\infty < \epsilon, \quad \boxed{\left\|\phi^1_\epsilon\right\|_{\mathcal{C}^2} <??}$$

$$\phi \in \mathcal{C}^\gamma \leftrightsquigarrow \quad \left\|\phi^1_\epsilon\right\|_{\mathcal{C}^2} \lesssim (1/\epsilon)^{2/\gamma}$$

$$\phi \in \log^q \leftrightsquigarrow \quad \left\|\phi^1_\epsilon\right\|_{\mathcal{C}^2} \lesssim e^{(1/\epsilon)^{1/q}}$$

We will need *summable errors* $\Rightarrow \epsilon = 1/j^2$

$\Rightarrow q > 2$: ϕ can be approximated with functions $\phi_j := \phi^1_{1/j}$ whose $\mathcal{C}^2$ norms diverge *sub-exponentially* in j

Idea of the method

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- study the sequence $\mathcal{L}^n/\lambda^n$ and prove *almost periodicity*
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Here

- We want to find λ *intrinsically*, as part of our method
- We just normalize (for g positive) by $\int \mathcal{L}^n g \,\mathrm{Leb}$, or $\min \mathcal{L}^n g$, and we will see the exponential behaviour later.

For simplicity: $\phi \in \mathcal{C}^2, g = \mathbb{1}$. Denote $\mathbb{1}_n := \mathcal{L}^n \mathbb{1}$ and $\mathbb{1}_n^* = \mathbb{1}_n / \min \mathbb{1}_n$.

Idea

We prove that $\max \mathbb{1}_n^* = \max \mathbb{1}_n / \min \mathbb{1}_n$ is bounded

Method: finding λ

Idea

We prove that $\max \mathbb{1}_n^* = \max \mathbb{1}_n / \min \mathbb{1}_n$ is bounded

Then:

$$\begin{cases} \max \mathbb{1}_{n+m} \leqslant \max \mathbb{1}_n \cdot \max \mathbb{1}_m \\ \min \mathbb{1}_{n+m} \geqslant \min \mathbb{1}_n \cdot \min \mathbb{1}_m \\ \max \mathbb{1}_n / \min \mathbb{1}_n \leqslant C \end{cases} \Rightarrow \boxed{\lambda := \inf_n (\max \mathbb{1}_n)^{1/n} := \sup_n (\min \mathbb{1}_n)^{1/n}}$$

To bound $\max / \min$, we bound $\Omega / \min = (\max - \min) / \min$

$$\boxed{\text{We need to bound } \Omega(\mathbb{1}_n^*)}$$

Bounding the oscillation ($k = 1$ for simplicity)

Bound on $dd^c \Rightarrow$ bound on oscillation

Lemma (Heuristic version)

$dd^c g \leqslant dd^c h$ then $\Omega(g, r) \lesssim \Omega(h, r)$.

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Lemma (More precise version)

- $\Omega(g, r) \lesssim \Omega(h, \sqrt{r}) + A\sqrt{r}$.
- *if $dd^c g_n \leqslant R$ with continuous potentials, then the family g_n is equicontinous.*

Here we want

$$dd^c \mathbb{1}_n^* \leqslant R$$

for some uniform R, for which we control the regularity of the potential

Bounding the oscillation of $\mathbb{1}_n^*$

Development of $\mathbb{1}_n$

$$dd^c \mathbb{1}_n = dd^c \left(\sum_{f^n(x)=y} e^{\phi + \phi(f(x)) + \ldots \phi(f^{n-\mathbf{1}}(x))} \mathbb{1} \right)$$

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Bounding the oscillation of $\mathbb{1}_n^*$

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$\vdots$ (more complicated with g, ϕ less regular)

$$dd^c \mathbb{1}_n^* \lesssim \sum_{j=0}^{n} \left(\frac{e^{\Omega(\phi)}}{d} \right)^j \Omega(1_{n-j}^*) \|\phi\|_{\mathcal{C}^2} f_*^{j-1} \text{Leb}$$

$$\lesssim \sum_{j=0}^{\infty} \left(\frac{e^{\Omega(\phi)}}{d} \right)^j f_*^{j-1} \text{Leb}$$

Ok for mass. We still need to estimate the oscillation of the potential of the RHS. But what is this potential?

(Dynamical) potentials

$$\mathrm{Leb} = \mu + dd^c u_0 \qquad f_*^j\,\mathrm{Leb} = \mu + dd^c u_j$$

- u_0 is the Green function, which is γ-Hölder.
- Up to a Hölder continuous function, the potential of

$$\sum_{j=0}^{\infty} \left(\frac{e^{\Omega(\phi)}}{d} \right)^j f_*^{j-1}\,\mathrm{Leb} \quad \text{is given by} \quad \sum_{j=0}^{n} \left(\frac{e^{\Omega(\phi)}}{d} \right)^j u_j$$

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Lemma

1. u_j *is* $\gamma/2^j$ *Hölder.*
2. $\|u_j\|_\infty \lesssim d^n/\delta^n$ *for all* $\delta < d$.

$$\Rightarrow \quad \sum_{j=0}^{\infty} \left(\frac{e^{\Omega(\phi)}}{d}\right) u_j \in \log^p \quad \forall p$$

$$\Rightarrow \quad \|\mathbb{1}_n^*\|_{\log^p} < C_p \quad \forall n, p$$

When ϕ is not $\mathcal{C}^2$

$$\phi \in \log^q \Rightarrow \begin{cases} \phi = \phi_j^1 + \phi_j^2 \\ \|\phi_j^2\|_\infty \leqslant 1/j^2 \\ \|\phi_j^1\|_{\mathcal{C}^2} \leqslant e^{j^{2/q}} \leftarrow \text{ sub-exponential} \end{cases}$$

$$dd^c \mathbb{1}_n^* \lesssim \sum_{j=1}^{n} \left(\frac{e^{\Omega(\phi)}}{d} \right)^j \Omega(1_{n-j}^*) \left\| \phi_j^1 \right\|_{\mathcal{C}^2} f_*^{j-1} \text{Leb}$$

$\Rightarrow$ Existence and uniqueness of equilibrium state and conformal measure for all ϕ with $\|\phi\|_{\log^q} < \infty$ for some $q > 2$ (Theorem 1)

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Second (and main) goal: find a norm so that this convergence becomes a contraction in a suitable space of functions.

Norm and spectral gap

First consider the case $\phi = 0$

DSH norm (Dinh-Sibony)

$\|g\|_{DSH} = \min \|R^+\|$, where $dd^c g = R^+ - R^-$, $R^\pm$ positive measures

Then

$$\left\| \frac{f_* g}{d} \right\|_{DSH} \leqslant \frac{1}{d} \|f_* R^+ - f_* R^-\| = \frac{1}{d} \|g\|_{DSH}$$

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Here, if we try to do the same

$$dd^c \mathcal{L}_\phi(g) \sim \sum_{f(x)=y} e^{\phi(x)} dd^c g + g dd^c \phi e^\phi + e^\phi \partial g \bar{\partial} \phi + e^\phi \partial \phi \bar{\partial} g$$

$$dd^c \mathcal{L}_\phi^n(g) \sim \ldots$$

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- The operator dd^c is complex (commutes with f_*). Here non complex perturbation ($f_*(e^\phi \cdot)$). No way to keep complex norm like *DSH* even if ϕ is smooth.
- Serious problem for norm is given by mixed terms.
- f_* does not work well with Hölder, so need weaker.

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Idea 0

Use a norm with bound on dd^c+ regularity; then obtain spectral gap on "real" norm by *interpolation*. We use *pluripotential theory* to study the norm with dd^c.

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Idea 1

Use something like $\|\cdot\|_p :\cong \|\cdot\|_{DSH} + \|\cdot\|_{\log^p}$ ($\|\nu\|_p :\cong \|\nu\|_* + \|u_\nu\|_{\log^p}$ for measures)

$\|\cdot\|_p :\cong \|\cdot\|_{DSH} + \|\cdot\|_{\log^p}$

Lemma

$$\left\|\partial g \wedge \bar{\partial} h\right\|_p \leqslant \|g\|_p \|h\|_p$$

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$\left\|\partial g \wedge \overline{\partial} h\right\|_p \leqslant \|g\|_p \|h\|_p$

... but other problems: loss of regularity!

$$\frac{dd^c \mathbb{1}_n}{\lambda^n} \lesssim \left(\frac{e^{\Omega(\phi)}}{d}\right)^n f_*^n dd^c g + \sum_{j=1}^{n} \left(\frac{e^{\Omega(\phi)}}{d}\right)^j \left\|\frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}}\right\|_\infty f_*^{j-1} dd^c \phi + \dots$$

The potential of the RHS is

$$\left(\frac{e^{\Omega(\phi)}}{d}\right)^n f_*^n g + \sum_{j=1}^{n} \left(\frac{e^{\Omega(\phi)}}{d}\right)^j \left\|\frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}}\right\|_\infty f_*^{j-1} \phi + \dots \in ???$$

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Lemma

$\left\|d^{-j} f_*^j \phi\right\|_{\log^p} \leqslant c_p(A) A^n \|\phi\|_{\log^p}$
for all $A > 1$

Theorem

$\|\phi\|_p < \infty$ *then* $\left\|f_*^j \phi / d^j\right\|_\infty \to 0$
exponentially (precise bounds).

$$\Rightarrow \left(\frac{e^{\Omega(\phi)}}{d}\right)^n f_*^n g + \sum_{j=1}^n \left(\frac{e^{\Omega(\phi)}}{d}\right)^j \left\|\frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}}\right\|_\infty f_*^j \phi \in \log^q \text{ for some explicit } q < p$$

$\Rightarrow$ convergence $\left\|\lambda^{-n} \mathcal{L}_\phi^n g\right\|_q \to 0$, uniform in g, but no spectral gap yet!

Idea 2: a dynamical norm

Definition

$$\|R\|_{\alpha,p} := \min c \colon R \leqslant c \sum_j \alpha^j f_*^j S \text{ for some } \|S\|_p \leqslant 1.$$

By *definition*: $\|f_* R\|_{\alpha,p} \leqslant \frac{1}{\alpha} \|R\|_{\alpha,p}$

$$\Rightarrow \left\| \left(\frac{e^{\Omega(\phi)}}{d} \right)^n f_*^n dd^c g + \sum_{j=1}^n \left(\frac{e^{\Omega(\phi)}}{d} \right)^j \left\| \frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}} \right\|_\infty f_*^{j-1} dd^c \phi \right\|_{\alpha,p}$$

$$\leqslant \left(\frac{e^{\Omega(\phi)}}{\alpha d} \right)^n \|dd^c g\|_{\alpha,p} + \sum_{j=1}^n \left(\frac{e^{\Omega(\phi)}}{\alpha d} \right)^j \left\| \frac{\mathcal{L}^{n-j} g}{\lambda^{n-j}} \right\|_\infty \|dd^c \phi\|_{\alpha,p}$$

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$$\leqslant c_n \|dd^c g\|_{\alpha,p} \to 0!$$

Where is the problem? We did not consider the mixed terms!

The problem of the mixed terms

Definition

$\|g\|_{\alpha,p} := \|dd^c g\| := \min c \colon dd^c g \leqslant c \sum_j \alpha^j f_*^j S$ for some $\|S\|_p \leqslant 1$.

Then we have

$$dd^c(gh) = g dd^c h + h dd^c g + i\partial g \wedge \overline{\partial} h + i\partial h \wedge \overline{\partial} g$$

$$\|dd^c gh\|_{\alpha,p} \leqslant \|g\|_\infty \|dd^c h\|_{\alpha,p} + \|h\|_\infty \|dd^c g\|_{\alpha,p} + \left\| i\partial g \wedge \overline{\partial} h + i\partial h \wedge \overline{\partial} g \right\|_{\alpha,p}$$

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$\|g\|_{\alpha,p} := \|dd^c g\| := \min c\colon dd^c g \leqslant c \sum_j \alpha^j f^j_* S$ for some $\|S\|_p \leqslant 1$.

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$$dd^c(gh) = gdd^c h + hdd^c g + i\partial g \wedge \overline{\partial} h + i\partial h \wedge \overline{\partial} g$$

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Modified definition

$\|g\|^2_{\alpha,p} := \left\| i\partial g \wedge \overline{\partial} g \right\| := \min c\colon i\partial g \wedge \overline{\partial} g \leqslant c \sum_j \alpha^j f^j_* S, \|S\|_p \leqslant 1.$

$$\|g\|^2_{\alpha,p} := \left\| i\partial g \wedge \overline{\partial} g \right\| := \min c \colon i\partial g \wedge \overline{\partial} g \leqslant c \sum_j \alpha^j f^j_* S \text{ for some } \|S\|_p \leqslant 1.$$

- ok for mixed terms
- still good shifting property (less direct)
- spectral gap

$$\|g\|^2_{\alpha,p} := \left\| i\partial g \wedge \overline{\partial} g \right\| := \min c \colon i\partial g \wedge \overline{\partial} g \leqslant c \sum_j \alpha^j f^j_* S \text{ for some } \|S\|_p \leqslant 1.$$

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Main issue... all the method was based on the

Lemma

$dd^c g \leqslant dd^c h \Rightarrow \Omega(g, r) \lesssim \Omega(h, r)$

We need

Lemma

$\partial g \wedge \overline{\partial} g \leqslant dd^c h \Rightarrow \Omega(g, r) \lesssim \Omega(h, r)$

$$\|g\|^2_{\alpha,p} := \left\|i\partial g \wedge \overline{\partial} g\right\| := \min c \colon i\partial g \wedge \overline{\partial} g \leqslant c \sum_j \alpha^j f^j_* S \text{ for some } \|S\|_p \leqslant 1.$$

- ok for mixed terms
- still good shifting property (less direct)
- spectral gap ? YES!

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Lemma

$\partial g \wedge \overline{\partial} g \leqslant dd^c h \Rightarrow \Omega(g,r) \lesssim \Omega(h,r)$

Much more involved but true!

Spectral gap(s) by interpolation

We have a spectral gap for the norm $\|g\|^2_{\alpha,p} := \left\|\partial g \wedge \overline{\partial} g\right\|_{\alpha,p}$
$\Rightarrow$ We build a γ-Hölder-like norm from $\|\cdot\|_{\alpha,p}$:

Definition

$$\|g\|_{\alpha,p,\gamma} := \min c \colon \forall 0 < \epsilon < 1 \colon \begin{cases} g = g^1_\epsilon + g^2_\epsilon \\ \|g^2_\epsilon\|_\infty \leqslant c\epsilon \\ \|g^1_\epsilon\|_{\alpha,p} \leqslant c(1/\epsilon)^{1/\gamma} \end{cases}$$

- $\log^q \leqslant \|\cdot\|_{\alpha,p,\gamma} \leqslant \|\cdot\|_{\mathcal{C}^\gamma}$
- Interpolation techniques (all the method is stable under "sub-exponential perturbations"):

Spectral gap for $\|\cdot\|_{\alpha,p,\gamma} \lesssim \mathcal{C}^\gamma$