Hausdorff dimension of Julia sets in the logistic family

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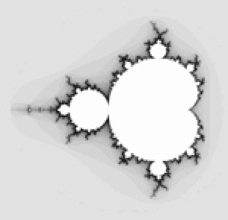
24th August, 2020

Hausdorff dimension of Julia sets

$f_c : z \mapsto z^2 + c$, Julia set J_c, its Hausdorff dimension $\mathrm{HD}(\mathcal{J}_c)$ is

- 1, if $c = 0$ (unit circle) or $c = -2$ (segment $[-2, 2]$) ;
- > 1 for all other $c \in \mathcal{M}$ (Zdunik) ;
- 
- < 2 if f_c is Collet-Eckmann (thus for harmonic measure almost every $c \in \partial\mathcal{M}$) (Przytycki, Rhode, Graczyk, Smirnov, Świątek).
- 2, dense set of $c \in \partial\mathcal{M}$ (Shishikura).

Real parameters $c \in \mathbb{R}$ near -2

$1 < \limsup_{c \to -2^+} \mathrm{HD}(\mathcal{J}_c) = \sup_{c \in \mathbb{R}} \mathrm{HD}(\mathcal{J}_c)$
(renormalisation argument)

$\mathrm{HD}(\mathcal{J}_c) - 1 \sim \sqrt{\varepsilon}$ if $c = -2 - \varepsilon$ (Jiang).

Real parameters $c \in \mathbb{R}$ with $c > -2$

Let $\mathcal{A} = \{c \geq -2 : \log |Df^n(c)| > n/2$ for all $n \geq 1\}$.
It is a set of (uniformly) Collet-Eckmann parameters.

Theorem (Jakobson ; Benedicks, Carleson)

-2 is a one-sided density point of $\mathcal{A}$.

Theorem (Graczyk, Smirnov)

$c \mapsto HD(\mathcal{J}_c)$ is continuous on $\mathcal{A}$.

Asymptotic behaviour ?

Techniques to estimate dimension

Poincaré exponent :

- all preimages of order n.

porosity :

- holes at all small scales.

beta numbers - wiggliness :

- local geometry at all small scales.
- $\beta(x, r)$.

beta numbers - mean wiggliness :

- average of local geometry over different scales.

Poincaré exponent

- Solving

$$\sum_{i=1}^{d} s_i^{-t} = 1$$

  gives the Hausdorff dimension of the invariant set of a map with d expanding branches $z \mapsto s_i z + a_i$.

- If the map φ is expanding, not necessarily piecewise-linear, one can look at

$$\sum_{y \in \varphi^{-n}(0)} |D\varphi^n(y)|^{-t}.$$

  If it tends to ∞ as $n \to \infty$, then HD $\geq t$.
  It it tends to 0, HD $\leq t$.

- Similarly for f_c, $c \in \mathcal{A}$ (unused today).

β-numbers

Let K be a continuum.

Theorem (Bishop, Jones 1997)

If $\beta(x,r) \geq \beta_0$ *for all* $x \in K$, $r > 0$, *then HD*$(K) \geq 1 + \mathrm{const}\beta_0^2$.

Mean-wiggly version by Graczyk, Jones, Mihalache of both upper and lower estimates.

Theorem (Graczyk, Jones, Mihalache 2012-18)

If $X \subset \mathbb{R}^d$, $d \geq 2$, *such that for all* $x \in X$,

$$\liminf_{r \to 0^+} \frac{\int_r^1 \beta^2(x,t)\frac{dt}{t}}{-\log r} \leq \beta_0^2,$$

then

$$HD(X) \leq 1 + C\beta_0^2,$$

C *is a universal constant.*

Julia set, large scale geometry

$$\mathcal{E}_a = \{z : |z-2| + |z+2| \le 4 + a\}.$$

Faulty upper bound heuristic

If $c = -2 + \varepsilon$, $\varepsilon > 0$, then the Julia set is contained in an ellipse of width $\sqrt{\varepsilon}$. If the dynamics is non-uniformly expanding, there is a large set $W \subset \mathcal{J}_c$ of points which go to the large scale infinitely often with bounded distortion. Pull back to get $\beta(x, r) \leq C\sqrt{\varepsilon}$ at most points x and most scales r. Then $\mathrm{HD}(J_c) \leq 1 + C'\varepsilon$.

$$\liminf_{r \to 0^+} \frac{\int_r^1 \beta^2(x, t)\, \frac{dt}{t}}{-\log r} \leq \beta_0^2,$$

Main result

Theorem (D, Graczyk, Mihalache, 2020)

For some constant $C > 1$ and for $f_c : z \mapsto z^2 + c$, $c + 2 = \varepsilon$,

$$1 + C^{-1}\sqrt{\varepsilon} \le HD(\mathcal{J}_c) \le 1 + C|\log(\varepsilon)|^{3/2}\sqrt{\varepsilon}$$

for all c from a subset of real Collet-Eckmann parameters with the point -2 as a one-sided Lebesgue density point.

Notes on the upper bound :

- Frequency estimate uses Graczyk-Smirnov, conformal measures etc ;
- $|\log(\varepsilon)|^{1/2}$ comes from
  $$\nu_c(B(c, \varepsilon)) \le C\sqrt{\varepsilon}|\log(\varepsilon)|^{1/2}.$$
- the remaining $|\log(\varepsilon)|$ is necessary (it seems).

Lower bound

Theorem (D, Graczyk, Mihalache 2020)

For $c+2=\varepsilon$, $\varepsilon>0$ small,

$$HD(J_c) \geq 1 + C\sqrt{\varepsilon}.$$

Note : all parameters...

- real estimates (interval map) ;
- Poincaré exponent estimates, $1-\varepsilon^{3/4}$;
- Lyapunox exponent, Chebyshev Inequality ;
- Poincaré exponent type estimates, $1+\rho\varepsilon^{1/2}$;
- complexify and add complex branch with derivative $\sim\sqrt{\varepsilon}$;
- Binomial expansion - Poincaré exponent estimates.

Induction for the interval map

$f_c : \mathbb{R} \to \mathbb{R}$, $c + 2 = \varepsilon > 0$.

There exists an expanding induced map φ with bounded distortion etc s.t.

- complement of domain has measure bounded by $C\varepsilon^{3/4}$;
- exponential tails $m(\{x : |D\varphi(x)| \geq e^t\}) \leq Ce^{\alpha t}$.

Poincaré exponent estimate

- complement of domain has measure bounded by $C\varepsilon^{3/4}$;
- exponential tails $m(\{x : |D\varphi(x)| \geq e^t\}) \leq Ce^{\alpha t}$.

Let

$$Q(n,t) = \sum_{y \in \varphi^{-n}(0)} |D\varphi^n(y)|^{-t}.$$

If $t = 1 - \theta\varepsilon^{3/4}$ then

$$|D\varphi^n(y)|^{-t} \geq 2^{n\theta\varepsilon^{3/4}} |D\varphi^n(y)|^{-1}.$$

Hence

$$Q(n,t) \geq (1 + \theta\varepsilon^{3/4} \log 2)^n (1 - C'\varepsilon^{3/4})^n.$$

For $\theta > 4C'$, $Q(n,t) \to \infty$ as $n \to \infty$ so $\mathrm{HD}(\Lambda) \geq t$.

Increasing the exponent

φ expanding induced map, invariant set Λ, $\text{HD}(\Lambda) \geq 1 - C''\varepsilon^{3/4}$.
$Q(n, 1 - 4C'\varepsilon^{3/4}) \geq 1$ for large n.

Lemma

Let $s = 1 + \rho\sqrt{\varepsilon}$. $Q(n,s) \geq C^{-1}(1 - \rho C\sqrt{\varepsilon})^n$.

We don't have a good estimate of the form
$|D\varphi^n|^{-s} \geq (1 - C\rho\sqrt{\varepsilon})^n |D\varphi^n(y)|^{-1}$ for all y.

- conformal measure ν, exponent $\text{HD}(\Lambda)$.
- equivalent invariant probability measure σ.
- Lyapunov exponent $\chi = \frac{1}{n} \int \log |D\varphi^n| \, d\sigma$ for all n.
- Chebyshev Inequality : $\sigma(\{y : \log |D\varphi^n| \leq 2n\chi\} \geq \frac{1}{2}$.

exponential tails estimate gives universal bound for χ.

Complexify and add complex branch

branches of φ extend to map univalently over $\mathbb{C} \setminus \overline{U}$.

complex branch has diameter $\sim \sqrt{\varepsilon}$

Reversing the binomial expansion

The resulting Cantor repeller (φ plus a complex branch ζ, derivative $\leq C\varepsilon^{-1/2}$) we denote by ψ. Let

$$S(n, s) = \sum_{\psi^{-n}(0)} |D\psi^n(y)|^{-s}.$$

$S(n, s) \to \infty$ if ρ is chosen small, so HD $\geq s = 1 + \rho\sqrt{\varepsilon}$.