Constructing examples of oscillating wandering domains

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joint work with Phil Rippon and Gwyneth Stallard

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On geometric complexity of Julia sets II
26 August, 2020
Outline

1. Introduction

2. Motivation
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3. Our construction
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Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function.
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**Definition 1**

Let $U$ be a Fatou component of $f$. If $f^n(U) \cap f^m(U) = \emptyset$, for all $m, n \in \mathbb{N}$, with $m \neq n$ then $U$ is a **wandering domain**.
Introduction II

- Baker was the first to give an example of a transcendental entire function with a wandering domain, which was multiply connected, in 1976.
- Since then several authors have constructed functions with simply connected wandering domains using techniques such as approximation theory, quasiconformal folding and quasiconformal surgery.

Let $U$ be a wandering domain of $f$. Then $U$ is of one of the following types:

- **escaping**, if $f^n(z) \to \infty$ for all $z \in U$;
- **oscillating**, if there exist $(n_k)$, $(m_k)$ such that $f^{n_k}(z) \to \infty$ and $(f^{m_k}(z))$ stays bounded for all $z \in U$;
- **bounded (orbit)** if $(f^n(z))$ stays bounded for all $z \in U$. 

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Eremenko and Lyubich were the first to give an example of an oscillating wandering domain in 1987. Their technique was based on Approximation Theory but more recently examples of functions with oscillating wandering domains have been constructed using quasiconformal folding and quasiconformal surgery (Bishop, Fagella Jarque and Lazebnik, Lazebnik, Martí-Pete and Shishikura and others).
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▶ sequences of discs and half-annuli;
▶ a model function which was constant on the half-annuli and a translation on the discs;
▶ an extended version of Runge’s Approximation Theorem.
The Eremenko-Lyubich construction II

The model map $\phi$ maps

- $D_0$ to a point in $D_1$
- $D_1$ into $Q_1$
- $Q_1$ to a point in $D_2$

$f(Q_m) \subset D_{m+1}$ and $f^m(D_m) \subset Q_m$

$D_0$ lies in a wandering domain which is oscillating.
The Eremenko-Lyubich construction - open questions

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The Eremenko-Lyubich construction - open questions

In the Eremenko-Lyubich example we don’t have information on

- the degree of $f$ on the wandering domains;

- the boundedness of the wandering domains.
Recently, we classified simply connected wandering domains in terms of the hyperbolic distances between iterates and also in terms of the behaviour of orbits in relation to the boundaries of the wandering domains (joint work with A.M. Benini, N. Fagella, P. Rippon and G. Stallard).

A simply connected wandering domain $U$ of $f$ is

1. **contracting** if for all pairs of points in $U$ the hyperbolic distance between their orbits tends to 0;

2. **semi-contracting** if for all but countably many pairs of points in $U$ the hyperbolic distance between their orbits decreases but does not tend to 0; or

3. **eventually isometric** if for all but countably many pairs of points in $U$ the hyperbolic distance between their orbits is eventually constant.
We classify simply connected wandering domains in terms of convergence to the boundary considering the Euclidean distance between an orbit and the boundary of the wandering domain. A simply connected wandering domain $U$ of $f$ is of one of the following types

(a) away if all orbits stay away from the boundary;
(b) bungee if all orbits have a subsequence staying away from the boundary and a subsequence converging to it;
(c) converging if all orbits converge to the boundary.
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The two classification theorems give rise to 9 possible types of simply connected wandering domains.
Classifying simply connected wandering domains III

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Using a technique based on Approximation Theory we show that there are examples of six types of oscillating wandering domains.

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Note that a sequence of oscillating wandering domains has a subsequence where the domains shrink, and so the Euclidean distance of points from the boundary tends to 0.
Main construction theorem I

Let \((b_n)_{n \geq 0}\) be a sequence of Blaschke products of corresponding degree \(d_n \geq 1\), let \((\alpha_n)_{n \geq 0}\) be a sequence of real numbers with \(\alpha_0 = 1\) and \(\alpha_{n+1}/\alpha_n \leq 1/6\). For \(n \geq 0\), let

\[
D_n = D(9n, \alpha_n), \quad \Delta_n = D(a_n, \alpha_n), \quad \Delta'_n = D(a_n, 2\alpha_n), \quad \text{where} \ a_n = 9n + 4\alpha_n,
\]

\[
G_n = D(\kappa_n, 1) \quad \text{and} \quad G'_n = D(\kappa_n, 5/4), \quad \text{where} \ \kappa_n = a_n + 3.
\]

We consider the function

\[
\phi(z) = \begin{cases} 
    z + 9, & z \in \overline{D_n}, \ n \geq 0, \\
    \frac{z-a_n}{\alpha_n} + \kappa_n, & z \in \overline{\Delta_n}, \ n \geq 0, \\
    \alpha_{n+1}b_n(z - \kappa_n) + 4\alpha_{n+1}, & z \in \overline{G'_n}, \ n \geq 0.
\end{cases}
\]
Main construction theorem II

Let $V_m = \phi^m(\Delta_0) = D(c_m, \rho_m)$, $m \geq 0$. Then for a suitable choice of $(\alpha_n)$ there exists a transcendental entire function $f$ having an orbit of bounded, simply connected, oscillating, wandering domains $U_m$ such that, for $m \geq 0$,
Let \( V_m = \phi^m(\Delta_0) = D(c_m, \rho_m), \ m \geq 0 \). Then for a suitable choice of \((\alpha_n)\) there exists a transcendental entire function \( f \) having an orbit of bounded, simply connected, oscillating, wandering domains \( U_m \) such that, for \( m \geq 0 \),

(i) \( \overline{V''_m} := \overline{D(c_m, r_m)} \subset U_m \subset D(c_m, R_m) := V'_m \), where \( 0 < r_m < \rho_m < R_m \) and \( r_m, R_m \to \rho_m \) as \( m \to \infty \);

(ii) \( |f(z) - \phi(z)| \to 0 \) uniformly on \( \overline{V'_m} \) as \( m \to \infty \);

(iii) \( f(9n) = \phi(9n) = 9(n + 1) \) and \( f'(9n) = \phi'(9n) = 1 \);

(iv) \( f : U_m \to U_{m+1} \) has degree \( q_{m+1} \), where \( q_m = d_n \) if \( V_m = \Delta_n \) for some \( n \geq 0 \), and \( q_m = 1 \) otherwise.
Existence criterion - one of the main tools

The proof of the Main construction theorem uses the following result.

Theorem 1 (Benini, E., Fagella, Rippon and Stallard)
Let $f$ be a transcendental entire function and suppose that there exist Jordan curves $\gamma_n$ and $\Gamma_n$, $n \geq 0$, a bounded domain $D$, a subsequence $n_k \to \infty$ and compact sets $L_k$ (associated with $\Gamma_{n_k}$) such that

(a) $\Gamma_n$ surrounds $\gamma_n$, for $n \geq 0$;
(b) for every $k, n, m \geq 0$, $m \neq n$ the sets $L_k, D, \Gamma_m$ are in $\text{ext } \Gamma_n$;
(c) $\gamma_{n+1}$ surrounds $f(\gamma_n)$, for $n \geq 0$;
(d) $f(\Gamma_n)$ surrounds $\Gamma_{n+1}$, for $n \geq 0$;
(e) $f(D \cup \bigcup_{k \geq 0} L_k) \subset D$;
(f) $\max\{\text{dist}(z, L_k) : z \in \Gamma_{n_k}\} = o(\text{dist}(\gamma_{n_k}, \Gamma_{n_k}))$ as $k \to \infty$.

Then there exists an orbit of simply connected wandering domains $U_n$ such that $\text{int } \gamma_n \subset U_n \subset \text{int } \Gamma_n$, for $n \geq 0$. Moreover, if there exists $z_n \in \text{int } \gamma_n$ such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind $d_n$ times around $f(z_n)$, then $f : U_n \to U_n + 1$ has degree $d_n$, for $n \geq 0$. 
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(a) $\Gamma_n$ surrounds $\gamma_n$, for $n \geq 0$;
(b) for every $k, n, m \geq 0$, $m \neq n$ the sets $L_k, \overline{D}, \Gamma_m$ are in $\text{ext} \Gamma_n$;
(c) $\gamma_{n+1}$ surrounds $f(\gamma_n)$, for $n \geq 0$;
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Then there exists an orbit of simply connected wandering domains $U_n$ such that $\overline{\text{int} \gamma_n} \subset U_n \subset \text{int} \Gamma_n$, for $n \geq 0$.

Moreover, if there exists $z_n \in \text{int} \gamma_n$ such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind $d_n$ times around $f(z_n)$, then $f : U_n \to U_{n+1}$ has degree $d_n$, for $n \geq 0$. 
Existence criterion figure
Main construction theorem - Sketch of proof I

Step 1: Define $\alpha_n, \gamma_m$ and $\Gamma_m$ so that $\gamma_{m+1}$ surrounds $\phi(\gamma_m)$ and $\phi(\Gamma_m)$ surrounds $\Gamma_{m+1}$, and also the reefs $L_n$ and error quantities $\varepsilon_m > 0$. 
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\begin{align*}
\text{Step 2: Apply an extended version of Runge's Approximation Theorem, which is used by Eremenko and Lyubich, in which a model map can be approximated on a sequence of disjoint compact sets and we can prescribe exactly the behaviour of $f$ and $f'$ at the centre of each $D_n$.}
\end{align*} 

\begin{align*}
\text{Step 3: Show that $\gamma_{m+1}$ surrounds $f(\gamma_m)$ (1) and $f(\Gamma_m)$ surrounds $\Gamma_{m+1}$. (2)}
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$$\gamma_{m+1} \text{ surrounds } f(\gamma_m)$$  \hspace{1cm} (1)

and

$$f(\Gamma_m) \text{ surrounds } \Gamma_{m+1}.$$  \hspace{1cm} (2)
Main construction theorem - Sketch of proof II

The main difficulty in this step is that although we are allowed to associate only one error quantity to each $D_n$, in fact we visit each such disc infinitely many times, every time mapping to a smaller disc closer to the centre of $D_n$. In other words, in order for (1) and (2) to hold we need to have a smaller error every time we visit $D_n$. 

Step 4: Apply Theorem 1 to show that $f$ has a sequence of bounded simply connected wandering domains, which by construction will be oscillating.
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We overcome this problem by choosing $\alpha_n$ carefully in Step 1 and by using a result by Eremenko and Lyubich which, provided that $f(9n) = 9(n+1) = \phi(9n)$ and $f'(9n) = 1$ gives us an estimate of how much smaller the actual error is as we get closer to $9_n$. 

\[
\begin{align*}
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Main construction - summary

- Choose a suitable collection of discs, some of them with radius 1, and some with shrinking radii, and a suitable model map $\phi$ which is basically either a translation or a translated contracted Blaschke product.
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- Apply Theorem 1.
Examples I

In order to construct an example of one of the 6 types we first need to choose a suitable Blaschke product. For the chosen Blaschke product we need to check whether it is of the required type.

Example 1

Let $b_n(z) = z^2$ for all $n \in \mathbb{N}$.

Then $B_n(0) = b_n \circ \cdots \circ b_1(0) = 0$, and $B_n(1/16) \to B_n(0)$ as $n \to \infty$, and so $\text{dist}_\mathbb{D}(B_n(0), B_n(1/16)) \to 0$ as $n \to \infty$. 
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Then \( B_n(0) = b_n \circ \cdots \circ b_1(0) = 0 \), and \( B_n(1/16) \to B_n(0) \) as \( n \to \infty \), and so \( \text{dist}_\mathbb{D}(B_n(0), B_n(1/16)) \to 0 \) as \( n \to \infty \).
We then apply the construction theorem to obtain a transcendental entire function \( f \) which has a sequence of wandering domains \((U_m)\) and proceed in 4 steps:

S1 Study the subsequences \( f^{m_n}(4) \) and \( f^{m_n}(4 + 1/16) \) which lie in \( G_n \). Prove that they are very close to \( \phi^{m_n}(4) \) and \( \phi^{m_n}(4 + 1/16) \) respectively.

S2 Deduce that \( f^{m_n}(4) \) stays away from the boundary of \( G_n \).

S3 Deduce that \( \text{dist}(f^{m_n}(4), f^{m_n}(4 + 1/16)) \to 0 \) as \( n \to \infty \), and so \( \text{dist}_{G_n}(f^{m_n}(4), f^{m_n}(4 + 1/16)) \to 0 \) as \( n \to \infty \).

S4 Step 2 implies that \( U_m \) is of bungee type and Step 3 implies that \( U_m \) is contracting.
Examples II

Note that we only need to look at the hyperbolic distance between the orbits of two points for the first classification and the orbit of one point for the second classification.
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Challenge: We cannot prescribe a whole orbit of a point since each orbit visits $D_n$ infinitely many times. Hence we need to estimate the errors for each and every example and show that they are so small that they don’t ‘destroy’ the properties of the orbit of the point under the model map.
Dziękuję