Constructing examples of oscillating wandering domains

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Outline

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   - The Eremenko-Lyubich example
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Introduction I

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function.

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Definition 1

Let U be a Fatou component of f. If $f^n(U) \cap f^m(U) = \emptyset$, for all $m, n \in \mathbb{N}$, with $m \neq n$ then U is a **wandering domain**.

Introduction II

- Baker was the first to give an example of a transcendental entire function with a wandering domain, which was multiply connected, in 1976.
- Since then several authors have constructed functions with simply connected wandering domains using techniques such as approximation theory, quasiconformal folding and quasiconformal surgery.

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Let U be a wandering domain of f. Then U is of one of the following types:

- **escaping**, if $f^n(z) \to \infty$ for all $z \in U$;
- **oscillating**, if there exist $(n_k), (m_k)$ such that $f^{n_k}(z) \to \infty$ and $(f^{m_k}(z))$ stays bounded for all $z \in U$;
- **bounded (orbit)** if $(f^n(z))$ stays bounded for all $z \in U$.

The Eremenko-Lyubich construction I

Eremenko and Lyubich were the first to give an example of an oscillating wandering domain in 1987. Their technique was based on Approximation Theory but more recently examples of functions with oscillating wandering domains have been constructed using quasiconformal folding and quasiconformal surgery (Bishop, Fagella Jarque and Lazebnik, Lazebnik, Martí-Pete and Shishikura and others).

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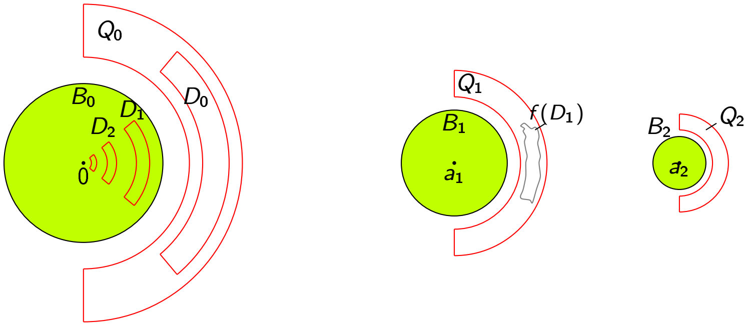
- sequences of discs and half-annuli;
- a model function which was constant on the half-annuli and a translation on the discs;
- an extended version of Runge's Approximation Theorem.

The Eremenko-Lyubich construction II

The model map ϕ maps

- D_0 to a point in D_1
- D_1 into Q_1
- Q_1 to a point in D_2



$$f(Q_m) \subset D_{m+1} \text{ and } f^m(D_m) \subset Q_m$$

D_0 lies in a wandering domain which is oscillating.

The Eremenko-Lyubich construction - open questions

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In the Eremenko-Lyubich example we don't have information on

- the degree of f on the wandering domains;
- the boundedness of the wandering domains.

Classifying simply connected wandering domains I

Recently, we classified simply connected wandering domains in terms of the hyperbolic distances between iterates and also in terms of the behaviour of orbits in relation to the boundaries of the wandering domains (joint work with A.M. Benini, N. Fagella, P. Rippon and G. Stallard).
A simply connected wandering domain U of f is

(1) **contracting** if for all pairs of points in U the hyperbolic distance between their orbits tends to 0;

(2) **semi-contracting** if for all but countably many pairs of points in U the hyperbolic distance between their orbits decreases but does not tend to 0; or

(3) **eventually isometric** if for all but countably many pairs of points in U the hyperbolic distance between their orbits is eventually constant.

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(a) **away** if all orbits stay away from the boundary;

(b) **bungee** if all orbits have a subsequence staying away from the boundary and a subsequence converging to it;

(c) **converging** if all orbits converge to the boundary.

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Using a technique based on Approximation Theory we show that there are examples of six types of oscillating wandering domains.

Classifications	away	bungee	converging
contracting	–	x	x
semi-contracting	–	x	x
eventually isometric	–	x	x

⚠ Note that a sequence of oscillating wandering domains has a subsequence where the domains shrink, and so the Euclidean distance of points from the boundary tends to 0.

Main construction theorem I

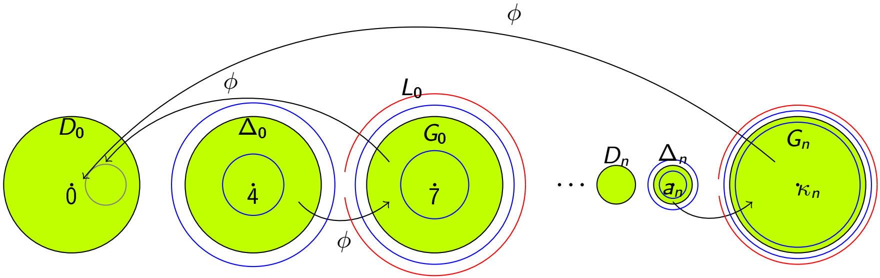
Let $(b_n)_{n\geq 0}$ be a sequence of Blaschke products of corresponding degree $d_n \geq 1$, let $(\alpha_n)_{n\geq 0}$ be a sequence of real numbers with $\alpha_0 = 1$ and $\alpha_{n+1}/\alpha_n \leq 1/6$. For $n \geq 0$, let

$$D_n = D(9n, \alpha_n), \Delta_n = D(a_n, \alpha_n), \Delta_n' = D(a_n, 2\alpha_n), \text{ where } a_n = 9n + 4\alpha_n,$$

$$G_n = D(\kappa_n, 1) \text{ and } G_n' = D(\kappa_n, 5/4), \text{ where } \kappa_n = a_n + 3.$$

We consider the function

$$\phi(z) = \begin{cases} z + 9, \ z \in \overline{D_n}, \ n \geq 0, \\ \frac{z - a_n}{\alpha_n} + \kappa_n, \ z \in \overline{\Delta_n'}, \ n \geq 0, \\ \alpha_{n+1} b_n(z - \kappa_n) + 4\alpha_{n+1}, \ z \in \overline{G_n'}, \ n \geq 0. \end{cases}$$



Main construction theorem II

Let $V_m = \phi^m(\Delta_0) = D(c_m, \rho_m)$, $m \geq 0$. Then for a suitable choice of (α_n) there exists a transcendental entire function f having an orbit of bounded, simply connected, oscillating, wandering domains U_m such that, for $m \geq 0$,

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(i) $\overline{V''_m} := \overline{D(c_m, r_m)} \subset U_m \subset D(c_m, R_m) := V'_m$, where $0 < r_m < \rho_m < R_m$ and $r_m, R_m \to \rho_m$ as $m \to \infty$;

(ii) $|f(z) - \phi(z)| \to 0$ uniformly on $\overline{V'_m}$ as $m \to \infty$;

(iii) $f(9n) = \phi(9n) = 9(n+1)$ and $f'(9n) = \phi'(9n) = 1$;

(iv) $f : U_m \to U_{m+1}$ has degree q_{m+1}, where $q_m = d_n$ if $V_m = \Delta_n$ for some $n \geq 0$, and $q_m = 1$ otherwise.

Existence criterion - one of the main tools

The proof of the Main construction theorem uses the following result.

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Theorem 1 (Benini, E., Fagella, Rippon and Stallard)

Let f be a transcendental entire function and suppose that there exist Jordan curves γ_n and Γ_n, $n \geq 0$, a bounded domain D, a subsequence $n_k \to \infty$ and compact sets L_k (associated with Γ_{n_k}) such that

(a) Γ_n *surrounds* γ_n, *for* $n \geq 0$;

(b) *for every* $k, n, m \geq 0$, $m \neq n$ *the sets* $L_k, \overline{D}, \Gamma_m$ *are in* $\operatorname{ext} \Gamma_n$;

(c) γ_{n+1} *surrounds* $f(\gamma_n)$, *for* $n \geq 0$;

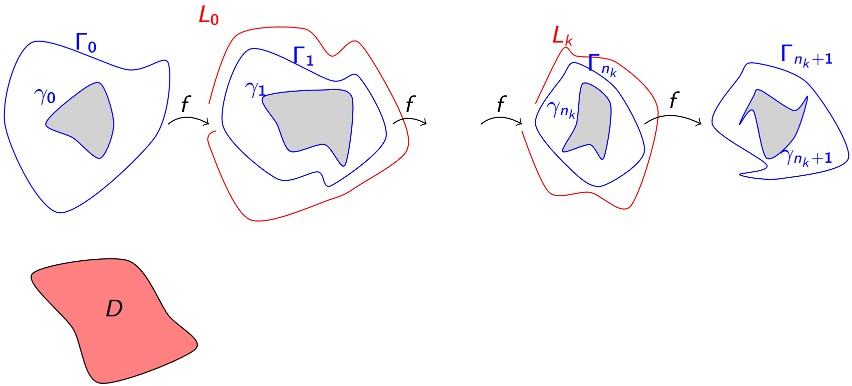
(d) $f(\Gamma_n)$ *surrounds* Γ_{n+1}, *for* $n \geq 0$;

(e) $f(\overline{D} \cup \bigcup_{k \geq 0} L_k) \subset D$;

(f) $\max\{\operatorname{dist}(z, L_k) : z \in \Gamma_{n_k}\} = o(\operatorname{dist}(\gamma_{n_k}, \Gamma_{n_k}))$ *as* $k \to \infty$.

Then there exists an orbit of simply connected wandering domains U_n such that $\overline{\operatorname{int} \gamma_n} \subset U_n \subset \operatorname{int} \Gamma_n$, *for* $n \geq 0$.
Moreover, if there exists $z_n \in \operatorname{int} \gamma_n$ such that both $f(\gamma_n)$ and $f(\Gamma_n)$ wind d_n times around $f(z_n)$, then $f : U_n \to U_{n+1}$ has degree d_n, for $n \geq 0$.

Existence criterion figure



Main construction theorem - Sketch of proof I

Step 1: Define α_n, γ_m and Γ_m so that γ_{m+1} *surrounds* $\phi(\gamma_m)$ and $\phi(\Gamma_m)$ *surrounds* Γ_{m+1}, and also the reefs L_n and error quantities $\varepsilon_m > 0$.

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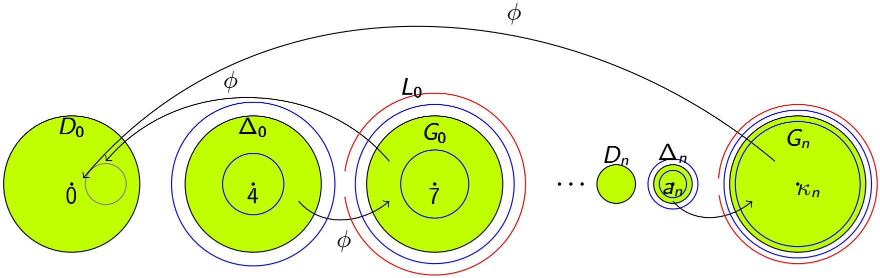

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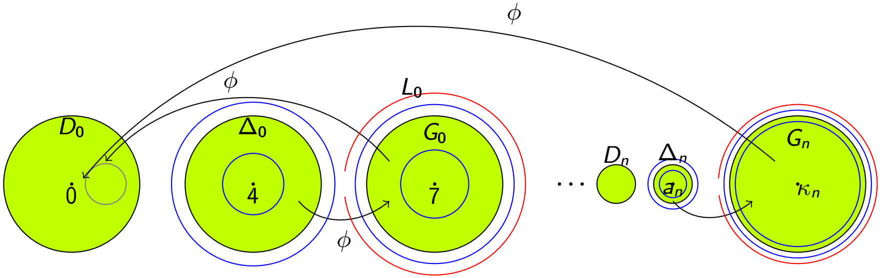

Step 2: Apply an extended version of Runge's Approximation Theorem, which is the one used by Eremenko and Lyubich, in which a model map can be approximated on a sequence of disjoint compact sets and we can prescribe *exactly* the behaviour of f and f' at the centre of each D_n.

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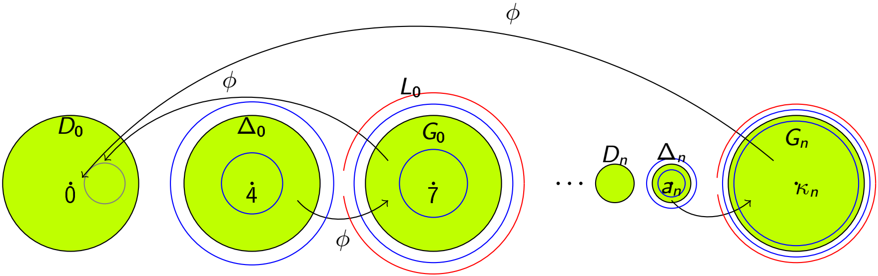

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Step 3: Show that

$$\gamma_{m+1} \ \textit{surrounds}\ f(\gamma_m) \tag{1}$$

and

$$f(\Gamma_m) \ \textit{surrounds}\ \Gamma_{m+1}. \tag{2}$$

Main construction theorem - Sketch of proof II

The main difficulty in this step is that although we are allowed to associate only one error quantity to each D_n, in fact we visit each such disc infinitely many times, every time mapping to a smaller disc closer to the centre of D_n. In other words, in order for (1) and (2) to hold we need to have a smaller error every time we visit D_n.

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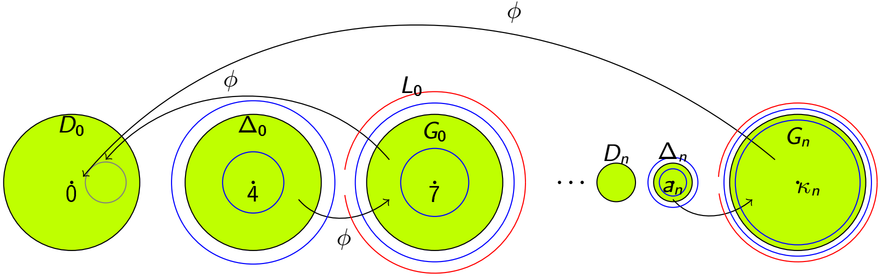
We overcome this problem by choosing α_n carefully in Step 1 and by using a result by Eremenko and Lyubich which, provided that $f(9n) = 9(n+1) = \phi(9_n)$ and $f'(9n) = 1$ gives us an estimate of how much smaller the actual error is as we get closer to 9_n.



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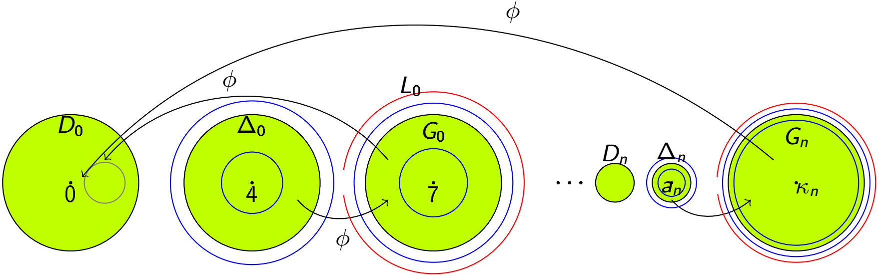

Step 4: Apply Theorem 1 to show that f has a sequence of bounded simply connected wandering domains, which by construction will be oscillating.

Main construction - summary

-  Choose a suitable collection of discs, some of them with radius 1, and some with shrinking radii, and a suitable model map ϕ which is basically either a translation or a translated contracted Blaschke product.

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-  Show that the error is as small as we need every time we visit D_n.
-  Apply Theorem 1.

Examples I

In order to construct an example of one of the 6 types we first need to choose a suitable Blaschke product. For the chosen Blaschke product we need to check whether it is of the required type.

Example 1

Let $b_n(z) = z^2$ for all $n \in \mathbb{N}$.
Then $B_n(0) = b_n \circ \cdots \circ b_1(0) = 0$, and $B_n(1/16) \to B_n(0)$ as $n \to \infty$, and so $\mathrm{dist}_{\mathbb{D}}(B_n(0), B_n(1/16)) \to 0$ as $n \to \infty$.

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We then apply the construction theorem to obtain a transcendental entire function f which has a sequence of wandering domains (U_m) and proceed in 4 steps:

- S1 Study the subsequences $f^{m_n}(4)$ and $f^{m_n}(4 + 1/16)$ which lie in G_n. Prove that they are very close to $\phi^{m_n}(4)$ and $\phi^{m_n}(4 + 1/16)$ respectively.
- S2 Deduce that $f^{m_n}(4)$ stays away from the boundary of G_n.
- S3 Deduce that $\operatorname{dist}(f^{m_n}(4), f^{m_n}(4 + 1/16)) \to 0$ as $n \to \infty$, and so $\operatorname{dist}_{G_n}(f^{m_n}(4), f^{m_n}(4 + 1/16)) \to 0$ as $n \to \infty$.
- S4 Step 2 implies that U_m is of bungee type and Step 3 implies that U_m is contracting.

Examples II

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Challenge: We cannot prescribe a whole orbit of a point since each orbit visits D_n infinitely many times. Hence we need to estimate the errors for each and every example and show that they are so small that they don't 'destroy' the properties of the orbit of the point under the model map.


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