The Permutable Mystery Tour

For transcendental meromorphic functions

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Outline

The Problem of Commuting Functions

The Problem

Given two analytic functions f and g, is it true that

$$f \circ g = g \circ f \Leftrightarrow J(f) = J(g) \ ?$$

The Problem of Commuting Functions

The Answer

- Polynomials
  - $\Rightarrow$ Fatou, 1920; Julia, 1922; Beardon, 1990
  - $\Leftarrow$ Beardon, 1990; Schmidt & Steinmetz, 1995
    - * Terms and conditions apply

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  $\Leftarrow$ Levin & Przytycki, 1997; Ye, 2015

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  $\Rightarrow$ Baker, 1984; Bergweiler & Hinkkanen, 1999; Benini, Rippon & Stallard, 2016

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- Transcendental entire functions
  - ⇒ Baker, 1984; Bergweiler & Hinkkanen, 1999; Benini, Rippon & Stallard, 2016
- Transcendental meromorphic functions
  - ⇒ Tsantaris, 2019; F, 2020

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Notation

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TEF	Transcendental entire function
TMF	Transcendental meromorphic (non-entire) function

The problem with escaping points

- The escaping set (Eremenko, 1989; Domínguez, 1998):

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  $\Rightarrow$ points in $I(f)$ escape at very different rates
- $I(f)$ is never empty, and is dense in the Julia set

Progress for Transcendental Entire Functions

Baker, 1958 & 1984

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Baker, 1984

Let f and g be commuting TEFs without escaping Fatou components. Then, $J(f) = J(g)$.

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Bergweiler & Hinkkanen, 1999

Define the fast escaping set as

$$A(f) := \{z : \exists\, \ell \in \mathbb{N} \text{ s.t. } |f^{n+\ell}(z)| \geq M^n(r, f) \text{ for all } n \in \mathbb{N}\}.$$

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BRS, 2016

If U is a multiply connected wandering domain of f and $f \circ g = g \circ f$, then $g(U)$ is also a multiply connected wandering domain of f.

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If U is a multiply connected wandering domain of f and $f \circ g = g \circ f$, then $g(U)$ is also a multiply connected wandering domain of f.

So, where are we now?

Let f and g be commuting transcendental entire functions without simply connected fast escaping wandering domains. Then, $J(f) = J(g)$.

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Rippon & Stallard, 2005

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  - $\gamma_n \to \infty$ as $n \to +\infty$
  - $\gamma_{n+1} \subset f(\gamma_n)$
  - Every component of $f^{-1}(E_{n+1})$ either lies in E_n or is surrounded by γ_1

$A(f)$ Via Outer Sequences

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Let f be a TMF with finitely many poles, and E_n an outer sequence for f. Then,

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- $A(f)$ is dense in the Julia set

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The Problem of Commuting Functions
Transcendental Functions and Escaping Points
Permutable TMFs
Revisiting $A(f)$
Finally, Permutable TMFs
Ping-pong Orbits

Hold Our Horses

Osborne & Sixsmith, 2016

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  - $\Rightarrow$ If f and g commute, they have the same poles
- A TMF f commutes with at most countably many other TMFs
- If a TMF commutes with a rational function R, then R is a Möbius transformation

The Plan

We need meromorphic versions of the following results:

Baker, 1984

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Let f and g be permutable TEFs. If $z \in \mathbb{C} \setminus A(f)$, then $g(z) \notin A(f)$.

The Outcome

F, 2020

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Let f and g be permutable TMFs. If U is a non-escaping Fatou component of f, then $g(U) \subset F(f)$.

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Let f and g be permutable TMFs with finitely many poles. If $z \in I(f) \setminus A(f)$, then $g(z) \notin A(f)$.

F, 2020

Let f and g be TMFs with finitely many poles s.t. $A(f) \subset J(f)$ and $A(g) \subset J(g)$. Then, $f \circ g = g \circ f \Rightarrow J(f) = J(g)$.

All Together Now

Tsantaris, 2019

Let f and g be permutable TMFs not in class $\mathcal{P}$. Then, $J(f) = J(g)$.

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$$\left(\text{Class } \mathcal{P}\text{: } f(z) = z_0 + \frac{e^{g(z)}}{(z - z_0)^m} \right)$$

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$$\left(\text{Class } \mathcal{P}\text{: } f(z) = z_0 + \frac{e^{g(z)}}{(z - z_0)^m}\right)$$

So, where are we now?

Let f and g be permutable TMFs. Then, $J(f) = J(g)$ except possibly when f and g are in class $\mathcal{P}$ and have simply connected fast escaping wandering domains.

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The Problem With Poles

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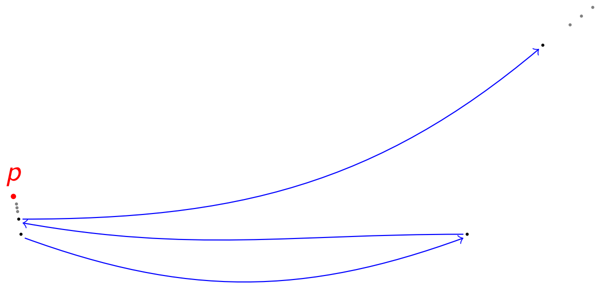
  - Then, if $z \notin I(f)$, we have $g(z) \notin I(f)$
- What about permutable TMFs?
  - If z is s.t. $f^{n_k}(z) \to p$, then it is possible that $g(z) \in I(f)$

The Problem With Poles



Ping-pong Orbits

Definition

We say that $z \in \mathbb{C}$ has a ping-pong orbit if there exist a pole p of f, subsequences $(f^{m_k})_{k\geq 1}$ and $(f^{n_k})_{k\geq 1}$ and a natural number $M \geq 1$ s.t.

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(i) $f^{m_k}(z) \to p$ and $f^{n_k}(z) \to \infty$;

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The set of all points with a ping-pong orbit is denoted by $BU_P(f)$.

Ping-pong Orbits

And where to find them

F, 2020

Let f be a TMF with finitely many poles. Then, $BU_P(f)$ is dense in $J(f)$. Furthermore, if $f \notin \mathcal{P}$, then points in $BU_P(f)$ can "escape" arbitrarily fast.

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F, 2020

There exist TMFs with a single pole, both in class $\mathcal{P}$ and outside of it, with ping-pong wandering domains.

* This theorem was brought to you by David Martí-Pete

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