The Permutable Mystery Tour
For transcendental meromorphic functions

Gustavo R. Ferreira

School of Mathematics and Statistics
The Open University

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Outline

The Problem of Commuting Functions
Transcendental Functions and Escaping Points

Permutable TMFs
Revisiting $A(f)$
Finally, Permutable TMFs
Ping-pong Orbits
The Problem of Commuting Functions

The Problem

Given two analytic functions $f$ and $g$, is it true that

$$f \circ g = g \circ f \Leftrightarrow J(f) = J(g)$$?
The Problem of Commuting Functions

The Answer

- **Polynomials**
  - \( \Rightarrow \) Fatou, 1920; Julia, 1922; Beardon, 1990
  - \( \Leftarrow \) Beardon, 1990; Schmidt & Steinmetz, 1995

* Terms and conditions apply
The Problem of Commuting Functions

The Answer

▶ Polynomials

⇒ Fatou, 1920; Julia, 1922; Beardon, 1990
⇐ Beardon, 1990; Schmidt & Steinmetz, 1995
* Terms and conditions apply

▶ Rational functions

⇒ Fatou, 1920; Julia, 1922; F, 2019
⇐ Levin & Przytycki, 1997; Ye, 2015
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The Problem of Commuting Functions

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- **Polynomials**
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- **Rational functions**
  - $\Rightarrow$ Fatou, 1920; Julia, 1922; F, 2019
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- **Transcendental entire functions**
  - $\Rightarrow$ Baker, 1984; Bergweiler & Hinkkanen, 1999; Benini, Rippon & Stallard, 2016
The Problem of Commuting Functions

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- Transcendental entire functions
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- Transcendental meromorphic functions
  - $\Rightarrow$ Tsantaris, 2019; F, 2020
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Transcendental Functions and Escaping Points

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Notation

\[ f, g \quad \text{Transcendental (entire or meromorphic) functions} \]
Notation

\begin{itemize}
  \item \( f, g \) \quad \text{Transcendental (entire or meromorphic) functions}
  \item \( M(r, f) \) \quad \text{The maximum modulus function}
  \item \( M(r, f) := \max\{|f(z)| : |z| = r\} \)
\end{itemize}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f, g$</td>
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<td>$M(r, f)$</td>
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</tr>
<tr>
<td>TEF</td>
<td>Transcendental entire function</td>
</tr>
<tr>
<td>TMF</td>
<td>Transcendental meromorphic (non-entire) function</td>
</tr>
</tbody>
</table>
The problem with escaping points

The escaping set (Eremenko, 1989; Domínguez, 1998):

\[ I(f) := \{ z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to +\infty \} \]
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- \( f \) is transcendental \( \Rightarrow \) infinity is an essential singularity
  \( \Rightarrow \) points in \( I(f) \) escape at very different rates
- \( I(f) \) is never empty, and is dense in the Julia set
Progress for Transcendental Entire Functions
Baker, 1958 & 1984

1958: If \( f \circ g = g \circ f \), then \( g(J(f)) \subset J(f) \)
Progress for Transcendental Entire Functions
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! It suffices to show that \( f \circ g = g \circ f \Rightarrow g(F(f)) \subset F(f) \)
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1984: If $f \circ g = g \circ f$ and $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$
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! It suffices to show that $f \circ g = g \circ f \Rightarrow g(F(f)) \subset F(f)$

1984: If $f \circ g = g \circ f$ and $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$

Baker, 1984
Let $f$ and $g$ be commuting TEFs without escaping Fatou components. Then, $J(f) = J(g)$. 
Define the **fast escaping set** as

\[
A(f) := \{z : \exists \ell \in \mathbb{N} \text{ s.t. } |f^{n+\ell}(z)| \geq M^n(r, f) \text{ for all } n \in \mathbb{N}\}.
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► \( A(f) \subset I(f) \)
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Then,

- \( A(f) \subset I(f) \)
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- Every Fatou component in \( A(f) \) is a wandering domain
Progress for Transcendental Entire Functions
Bergweiler & Hinkkanen, 1999

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Then,

- $A(f) \subset I(f)$
- $A(f)$ is dense in $J(f)$
- Every Fatou component in $A(f)$ is a wandering domain
- If $f \circ g = g \circ f$, then $g^{-1}(A(f)) \subset A(f)$
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\[ \text{If } f \circ g = g \circ f, \text{ then } g^{-1}(A(f)) \subset A(f) \]

Bergweiler & Hinkkanen, 1999

Let \( f \) and \( g \) be commuting TEFs without fast escaping wandering domains. Then, \( J(f) = J(g) \).
BRS, 2016

If $U$ is a multiply connected wandering domain of $f$ and $f \circ g = g \circ f$, then $g(U)$ is also a multiply connected wandering domain of $f$. 
If $U$ is a multiply connected wandering domain of $f$ and $f \circ g = g \circ f$, then $g(U)$ is also a multiply connected wandering domain of $f$.

**So, where are we now?**

Let $f$ and $g$ be commuting transcendental entire functions without simply connected fast escaping wandering domains. Then, $J(f) = J(g)$. 

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The Problem of Commuting Functions
Transcendental Functions and Escaping Points

Permutable TMFs
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Ping-pong Orbits
Let $f$ be a transcendental meromorphic function with finitely many poles.
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- Let $\gamma$ be a Jordan curve. Its outer set is defined as its unbounded complementary component.
Outer Sequences
Rippon & Stallard, 2005

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- **Outer sequence for $f$:** Jordan curves $\gamma_n$ with outer sets $E_n$ s.t.
  - Every $\gamma_n$ surrounds all the poles of $f$
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- Outer sequence for $f$: Jordan curves $\gamma_n$ with outer sets $E_n$ s.t.
  - Every $\gamma_n$ surrounds all the poles of $f$
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  - $\gamma_{n+1} \subset f(\gamma_n)$
Let $f$ be a transcendental meromorphic function with finitely many poles.

- Let $\gamma$ be a Jordan curve. Its outer set is defined as its unbounded complementary component.
- **Outer sequence for $f$:** Jordan curves $\gamma_n$ with outer sets $E_n$ s.t.
  - Every $\gamma_n$ surrounds all the poles of $f$
  - $\gamma_n \to \infty$ as $n \to +\infty$
  - $\gamma_{n+1} \subset f(\gamma_n)$
  - Every component of $f^{-1}(E_{n+1})$ either lies in $E_n$ or is surrounded by $\gamma_1$
Let \( f \) be a TMF with finitely many poles, and \( E_n \) an outer sequence for \( f \). Then,

\[
A(f) := \{ z : \exists \ell \in \mathbb{N} \text{ s.t. } f^{n+\ell}(z) \in E_n \text{ for all } n \in \mathbb{N} \}.
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Let $f$ be a TMF with finitely many poles, and $E_n$ an outer sequence for $f$. Then,

$$A(f) := \{ z : \exists \ell \in \mathbb{N} \text{ s.t. } f^{n+\ell}(z) \in E_n \text{ for all } n \in \mathbb{N} \}.$$

$A(f)$ is independent of the choice of outer sequence.
A(f) Via Outer Sequences
Rippon & Stallard, 2005

Let \( f \) be a TMF with finitely many poles, and \( E_n \) an outer sequence for \( f \). Then,

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▶ \( A(f) \) is independent of the choice of outer sequence
▶ If \( f \) is entire, this definition agrees with the previous one
Let $f$ be a TMF with finitely many poles, and $E_n$ an outer sequence for $f$. Then,

$$A(f) := \{ z : \exists \ell \in \mathbb{N} \text{ s.t. } f^{n+\ell}(z) \in E_n \text{ for all } n \in \mathbb{N} \}.$$ 

- $A(f)$ is independent of the choice of outer sequence
- If $f$ is entire, this definition agrees with the previous one
- $A(f)$ is dense in the Julia set
The Permutable Mystery Tour

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Outline

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Transcendental Functions and Escaping Points

Permutable TMFs
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Finally, Permutable TMFs
Ping-pong Orbits
Hold Our Horses
Osborne & Sixsmith, 2016

- TMFs $f$ and $g$ commute iff, for every $z \in \mathbb{C}$, either $f(g(z)) = g(f(z))$ or neither side is defined.
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⇒ If $f$ and $g$ commute, they have the same poles.
Hold Our Horses
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  - $\Rightarrow$ If $f$ and $g$ commute, they have the same poles
- A TMF $f$ commutes with at most countably many other TMFs
Hold Our Horses
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- TMFs $f$ and $g$ commute iff, for every $z \in \mathbb{C}$, either $f(g(z)) = g(f(z))$ or neither side is defined.
  \[ \Rightarrow \] If $f$ and $g$ commute, they have the same poles
- A TMF $f$ commutes with at most countably many other TMFs
- If a TMF commutes with a rational function $R$, then $R$ is a Möbius transformation
The Plan

We need meromorphic versions of the following results:

Baker, 1984
Let $f$ and $g$ be permutable TEFs. If $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$. 
The Plan

We need meromorphic versions of the following results:

**Baker, 1984**
Let $f$ and $g$ be permutable TEFs. If $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$.

**Bergweiler & Hinkkanen, 1999**
Let $f$ and $g$ be permutable TEFs. If $z \in \mathbb{C} \setminus A(f)$, then $g(z) \notin A(f)$. 
Let $f$ and $g$ be permutable TMFs. If $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$. 

F, 2020
The Outcome

F, 2020

Let $f$ and $g$ be permutable TMFs. If $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$.

F, 2020

Let $f$ and $g$ be permutable TMFs with finitely many poles. If $z \in l(f) \setminus A(f)$, then $g(z) \notin A(f)$. 
The Outcome

F, 2020

Let $f$ and $g$ be permutable TMFs. If $U$ is a non-escaping Fatou component of $f$, then $g(U) \subset F(f)$.

F, 2020

Let $f$ and $g$ be permutable TMFs with finitely many poles. If $z \in I(f) \setminus A(f)$, then $g(z) \not\in A(f)$.

F, 2020

Let $f$ and $g$ be TMFs with finitely many poles s.t. $A(f) \subset J(f)$ and $A(g) \subset J(g)$. Then, $f \circ g = g \circ f \Rightarrow J(f) = J(g)$. 
Tsantaris, 2019

Let $f$ and $g$ be permutable TMFs not in class $\mathcal{P}$. Then, $J(f) = J(g)$. 

Tsantaris, 2019

Let $f$ and $g$ be permutable TMFs not in class $\mathcal{P}$. Then, $J(f) = J(g)$.

\[
\begin{align*}
\left( \text{Class } \mathcal{P}: f(z) &= z_0 + \frac{e^{g(z)}}{(z - z_0)^m} \right)
\end{align*}
\]
All Together Now

Tsantaris, 2019

Let $f$ and $g$ be permutable TMFs not in class $\mathcal{P}$. Then, $J(f) = J(g)$.

\[
\left(\text{Class } \mathcal{P}: f(z) = z_0 + \frac{e^{g(z)}}{(z - z_0)^m}\right)
\]

So, where are we now?

Let $f$ and $g$ be permutable TMFs. Then, $J(f) = J(g)$ except possibly when $f$ and $g$ are in class $\mathcal{P}$ and have simply connected fast escaping wandering domains.
Outline

The Problem of Commuting Functions
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Permutable TMFs
Revisiting $A(f)$
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Ping-pong Orbits
The Problem With Poles

Let $f$ and $g$ be permutable TEFs
The Problem With Poles

- Let $f$ and $g$ be permutable TEFs
  - Then, if $z \notin I(f)$, we have $g(z) \notin I(f)$
The Problem With Poles

- Let $f$ and $g$ be permutable TEFs
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- What about permutable TMFs?
The Problem With Poles

- Let $f$ and $g$ be permutable TEFs.
  - Then, if $z \notin I(f)$, we have $g(z) \notin I(f)$.
- What about permutable TMFs?
  - If $z$ is s.t. $f^{nk}(z) \to p$, then it is possible that $g(z) \in I(f)$. 
The Problem With Poles
Ping-pong Orbits

Definition

We say that $z \in \mathbb{C}$ has a ping-pong orbit if there exist a pole $p$ of $f$, subsequences $(f^{m_k})_{k \geq 1}$ and $(f^{n_k})_{k \geq 1}$ and a natural number $M \geq 1$ s.t.
Ping-pong Orbits

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(i) $f^{m_k}(z) \to p$ and $f^{n_k}(z) \to \infty$;
Ping-pong Orbits

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(i) $f^{m_k}(z) \to p$ and $f^{n_k}(z) \to \infty$;

(ii) $|m_k - n_k| \leq M$ and $|n_k - m_{k+1}| \leq M$. 
Ping-pong Orbits

Definition

We say that \( z \in \mathbb{C} \) has a ping-pong orbit if there exist a pole \( p \) of \( f \), subsequences \( (f^{m_k})_{k \geq 1} \) and \( (f^{n_k})_{k \geq 1} \) and a natural number \( M \geq 1 \) s.t.

1. \( f^{m_k}(z) \to p \) and \( f^{n_k}(z) \to \infty \);
2. \( |m_k - n_k| \leq M \) and \( |n_k - m_{k+1}| \leq M \).

The set of all points with a ping-pong orbit is denoted by \( BU_P(f) \).
Ping-pong Orbits
And where to find them

F, 2020
Let $f$ be a TMF with finitely many poles. Then, $BUP(f)$ is dense in $J(f)$. Furthermore, if $f \notin \mathcal{P}$, then points in $BUP(f)$ can “escape” arbitrarily fast.
Ping-pong Orbits
And where to find them

F, 2020

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F, 2020

There exist TMFs with a single pole, both in class $\mathcal{P}$ and outside of it, with ping-pong wandering domains.

* This theorem was brought to you by David Martí-Pete
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G.R. Ferreira
Escaping points of commuting meromorphic functions with finitely many poles.