Accumulation set of critical points of the multipliers in the quadratic family

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Multipliers as functions of the parameter

- Quadratic family: \( \{ f_c(z) = z^2 + c \mid c \in \mathbb{C} \} \).
- \( \mathcal{O} = \langle z_0, z_1, \ldots, z_{k-1} \rangle \) is a periodic orbit of period \( k \) for \( f_{c_0} \).
- **Notation:** \( |\mathcal{O}| := k \) – the period of \( \mathcal{O} \).
- Multiplier map: \( \rho_{\mathcal{O}}(c) = f'_c(z_0)f'_c(z_1) \cdots f'_c(z_{k-1}) \)

- \( \rho_{\mathcal{O}} \) is a locally analytic function around \( c_0 \), whenever \( \mathcal{O} \) is not a primitive parabolic orbit of \( f_{c_0} \).
- \( \rho_{\mathcal{O}} \) extends to a multiple-valued algebraic function on \( \mathbb{C} \).
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**Question:** What can we say about critical points and critical values of the multiplier maps (i.e., when \( \rho'_\mathcal{O}(c) = 0 \))? 

**Theorem (Sullivan, Douady-Hubbard):** The multiplier \( \rho_\mathcal{O} \) of an attracting periodic orbit is a Riemann mapping of the corresponding hyperbolic component.

\[ \rho^{-1}_\mathcal{O}: \mathbb{D} \to H \] is a conformal isomorphism.
Critical values of the multipliers

\[ \rho^{-1}_O : \mathbb{D} \to H \] is a conformal isomorphism.

**Observation:** If \( \rho^{-1}_O \) can be extended univalently to a fixed neighborhood \( U \ni \mathbb{D} \), then Koebe Distortion Theorem provides bounds on the shape of \( H \).
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\[ \rho^{-1}_O : \mathbb{D} \to H \] is a conformal isomorphism.

**Observation:** If \( \rho^{-1}_O \) can be extended univalently to a fixed neighborhood \( U \supset \mathbb{D} \), then Koebe Distortion Theorem provides bounds on the shape of \( H \).

Critical values of \( \rho_O \) are the only obstacles for an analytic extension of \( \rho^{-1}_O \) beyond \( \mathbb{D} \).
$\rho^{-1}_O : \mathbb{D} \to H$ is a conformal isomorphism.

**Observation:** If $\rho^{-1}_O$ can be extended univalently to a fixed neighborhood $U \ni \mathbb{D}$, then Koebe Distortion Theorem provides bounds on the shape of $H$.

Critical values of $\rho_O$ are the only obstacles for an analytic extension of $\rho^{-1}_O$ beyond $\mathbb{D}$.

**Problem:** Does there exist a neighborhood $U \ni \mathbb{D}$, such that $\rho^{-1}_O$ is univalent in $U$, for any periodic orbit $O$?

**Theorem (Levin 2009, Dezotti 2011):** $\rho^{-1}_O$ is univalent in $U_k \ni \mathbb{D}$, where $\partial U_k \cap \partial \mathbb{D} = \{1\}$, and $U_k$ depends on $k = |O|$. 

Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 3$

Number of critical points $= 2$
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 4$

Number of critical points $= 6$
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 5$

Number of critical points $= 20$
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 6$

Number of critical points $= 38$
Critical points of the multiplier maps $\rho_\mathcal{O}$, $|\mathcal{O}| = 7$

Number of critical points $= 102$
Critical points of the multiplier maps $\rho_\mathcal{O}, \quad |\mathcal{O}| = 8$

Number of critical points $= 198$
Minimal critical values of the multiplier maps $\rho_0$
Equidistribution of critical points of the multipliers

For any $s \in \mathbb{C}$ and any $k \in \mathbb{N}$,

$X_{s,k} := \{ c \in \mathbb{C} \mid \rho'_{\mathcal{O}}(c) = s, \text{ for some periodic orbit } \mathcal{O} \}.$

(Points in $X_{s,k}$ are counted with multiplicity.)

$\nu_{s,k} := \frac{1}{\# X_{s,k}} \sum_{c \in X_{s,k}} \delta_c.$

**Equidistribution Theorem (Firsova, G. 2019):** For every sequence of complex numbers $\{s_k\}_{k \in \mathbb{N}}$, such that

$$\limsup_{k \to +\infty} \frac{1}{k} \log |s_k| \leq \log 2,$$

the sequence of measures $\{\nu_{s_k,k}\}_{k \in \mathbb{N}}$ converges to $\mu_{\text{bif}}$ in the weak sense of measures on $\mathbb{C}$, as $k \to \infty$. 
Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any $\rho_0 \in \mathbb{C}$, the set of parameters $c$ (counted with multiplicity), such that $\rho_0(c) = \rho_0$, for some $\mathcal{O}$ of period $k$, equidistributes on the boundary of $\mathbb{M}$, as $k \to \infty$.

**Accumulation set:** For an infinite collection of points $A \subset \mathbb{C}$ (counted with multiplicities), its accumulation set consists of all points $z \in \mathbb{C}$, whose arbitrary neighborhood contains infinitely many points from $A$. 
Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any \( \rho_0 \in \mathbb{C} \), the set of parameters \( c \) (counted with multiplicity), such that \( \rho_0(\mathcal{O}(c)) = \rho_0 \), for some \( \mathcal{O} \) of period \( k \), equidistributes on the boundary of \( \mathbb{M} \), as \( k \to \infty \).

Accumulation set: For an infinite collection of points \( A \subset \mathbb{C} \) (counted with multiplicities), its accumulation set consists of all points \( z \in \mathbb{C} \), whose arbitrary neighborhood contains infinitely many points from \( A \).

\( X \subset \mathbb{C} \) is the accumulation set of critical points of the multipliers.

Theorem (Firsova, G. 2020): The accumulation set \( X \) is bounded, path connected and contains the Mandelbrot set \( \mathbb{M} \). Furthermore, the set \( X \setminus \mathbb{M} \) is nonempty and has a nonempty interior, and every critical point of any multiplier is in \( X \).
The accumulation set $\mathcal{X}$
Roots of the multipliers and Lyapunov exponents

The root of the multiplier of a periodic orbit \( \mathcal{O} \):

\[
g_{\mathcal{O}}(c) := \left[ \rho_{\mathcal{O}}(c) \right]^{1/|\mathcal{O}|}
\]

The Lyapunov exponent of an arbitrary orbit \( z_0, z_1, z_2, \ldots \):

\[
\lambda_c(z_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f'_c(z_j)|
\]

For a periodic orbit \( \mathcal{O} = \langle z_0, \ldots, z_{k-1} \rangle \),

\[
\lambda_c(z_0) = \log |g_{\mathcal{O}}(c)|
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Roots of the multipliers and Lyapunov exponents

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For a periodic orbit $\mathcal{O} = \langle z_0, \ldots, z_{k-1} \rangle$,

$$\lambda_c(z_0) = \log |g_{\mathcal{O}}(c)|$$

**Ergodic Theorem:** There exists a function $L : \mathbb{C} \to \mathbb{R}$, such that $\lambda_c(z_0) = L(c)$, for a.e. $z_0 \in J_c$ with respect to the harmonic measure on $J_c$.

**Przytycki’s formula:** $L(c) = \log 2 + \frac{1}{2} G_M(c)$. 
Roots of the multiplier maps in $\mathbb{C} \setminus \mathcal{M}$

- $\Omega_c^k$ is the set of all period $k$ cycles of $f_c$, for $c \in \mathbb{C}$.

**Lemma:** For any $\delta > 0$ and a compact subset $K \subset \mathbb{C} \setminus \mathcal{M}$, the following holds:

$$\lim_{k \to \infty} \frac{\# \{ \mathcal{O} \in \Omega_c^k : \| \log |g_\mathcal{O}| - L \|_K < \delta \} \# \Omega_c^k}{\# \Omega_c^k} = 1$$

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**Critical points of the multiplier map**
\( \Omega^k_c \) is the set of all period \( k \) cycles of \( f_c \), for \( c \in \mathbb{C} \).

**Lemma:** For any \( \delta > 0 \) and a compact subset \( K \subset \mathbb{C} \setminus M \), the following holds:

\[
\lim_{k \to \infty} \frac{\# \{ \mathcal{O} \in \Omega^k_c : \| \log |g_{\mathcal{O}}| - L \|_K < \delta \}}{\# \Omega^k_c} = 1
\]

or equivalently,

\[
\lim_{k \to \infty} \frac{\# \{ \mathcal{O} \in \Omega^k_c : \| g_{\mathcal{O}} - 2 \sqrt{\phi_M} \|_K < \delta \}}{\# \Omega^k_c} = 1,
\]

where

\[ \phi_M : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{D} \] is a conformal diffeomorphism.
The sets $\mathcal{Y}_c$

- $\Omega_c$ is the set of all repelling periodic orbits of $f_c$.
- For every $\mathcal{O} \in \Omega_{c_0}$, the function
  \[ \nu_{\mathcal{O}}(c) := \frac{\rho'_{\mathcal{O}}(c)}{|\mathcal{O}| \rho_\mathcal{O}(c)} = \left[\log g_{\mathcal{O}}(c)\right]' \]
  is defined and analytic around $c = c_0$.
- For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by
  \[ \mathcal{Y}_c := \{ \nu_{\mathcal{O}}(c) \mid \mathcal{O} \in \Omega_c \} \]
The sets $\mathcal{Y}_c$

- $\Omega_c$ is the set of all repelling periodic orbits of $f_c$.
- For every $O \in \Omega_{c_0}$, the function
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  \nu_O(c) := \frac{\rho'_O(c)}{|O| \rho_O(c)} = [\log g_O(c)]'
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- For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by
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  \mathcal{Y}_c := \{ \nu_O(c) \mid O \in \Omega_c \}.
  \]

Theorem (Firsova, G. 2020): The following two properties hold:

(i) For every parameter $c \in \mathbb{C} \setminus \{-2\}$, the set $\mathcal{Y}_c$ is convex; for $c = -2$, the set $\mathcal{Y}_{-2}$ is the union of a convex set and the point $-\frac{1}{6}$.

(ii) For every parameter $c \in \mathbb{C} \setminus \mathcal{M}$, the set $\mathcal{Y}_c$ is bounded. A parameter $c \in \mathbb{C} \setminus \mathcal{M}$ belongs to $\mathcal{X}$, if and only if $0 \in \mathcal{Y}_c$. 
Critical points of the Hausdorff dimension function

Hausdorff dimension function: \( \delta(c) := \dim_H(J_c) \)

Theorem (Bowen): The function \( \delta \) is real-analytic in each hyperbolic component (including the complement of \( \mathbb{M} \)).

Theorem (Y. M. He, H. Nie 2020): (Version for the quadratic family) If \( c \in \mathbb{C} \) is a hyperbolic parameter and \( 0 \notin \mathcal{V}_c \), then \( c \) is not a critical point of the function \( \delta \).

Corollary: The Hausdorff dimension function \( \delta \) has no critical points in \( \mathbb{C} \setminus \mathcal{X} \).
**Proof of (i): Averaging Lemma**

**Averaging Lemma:** Let $\mathcal{O}_1, \mathcal{O}_2$ be two distinct non-exceptional repelling periodic orbits of $f_c$. Then for any $t \in [0, 1]$, there exists a sequence of periodic orbits $\mathcal{O}_3, \mathcal{O}_4, \ldots$ of $f_c$, such that

$$g_{\mathcal{O}_j} \to g_{\mathcal{O}_1}^t g_{\mathcal{O}_2}^{1-t}, \quad \text{and} \quad \nu_{\mathcal{O}_j} \to t\nu_{\mathcal{O}_1} + (1-t)\nu_{\mathcal{O}_2}$$

uniformly on a neighborhood of $c$ for appropriate branches of the powers.
Proof of (ii)

\[ \nu_O(c) := \frac{\rho'_O(c)}{|O| \rho_O(c)} = [\log g_O(c)]' \]

**Lemma:** Let \( U \subset \mathbb{C} \setminus \partial M \) be an open domain and fix \( c \in U \). Then each map from the family

\[ \mathcal{F}_c = \{ \nu_O \mid O \in \Omega_c \} \]

is defined in \( U \), and \( \mathcal{F}_c \) is normal in \( U \).

**Corollary:** For every \( c \in \mathbb{C} \setminus \partial M \), we have

\[ \mathcal{Y}_c = \{ \nu(c) \mid \nu \in \overline{\mathcal{F}_c} \}. \]

**Proof of (ii):** \( c \in \mathcal{X} \iff \exists \) a sequence of points \( c_j \to c \) and a sequence of orbits \( O_j \in \Omega_c \), such that \( \nu_{O_j}(c_j) = 0 \), \( \implies \exists \) a map \( \nu \in \overline{\mathcal{F}_c} \), such that \( \nu(c) = 0 \iff 0 \in \mathcal{Y}_c \).

“\( \iff \)” also holds if \( \nu \not\equiv 0 \).

**Lemma:** If \( c \in \mathbb{C} \setminus \overline{M} \), then \( 0 \notin \overline{\mathcal{F}_c} \).
\( \mathcal{X} \) is bounded and path connected

**Lemma:** The set \( \mathcal{X} \) is bounded.

*Idea of the proof:* Normality of the family \( \{g_\mathcal{O} \mid \mathcal{O} \in \Omega_c\} \) in \( \mathbb{C} \setminus \mathcal{M} \).

**Lemma:** The set \( \mathcal{X} \cup \mathcal{M} \) is path connected.

*Idea of the proof:*

- Let \( c_0 \in \mathcal{X} \setminus \mathcal{M} \) and \( \nu_\mathcal{O}(c_0) = 0 \), for some orbit \( \mathcal{O} \). Let \( \mathcal{O}' \) be another orbit, and consider

  \[
  \nu_t := (1 - t)\nu_\mathcal{O} + t\nu_{\mathcal{O}'}, \quad \text{for } t \in [0, 1].
  \]

  Then the curve

  \[
  [0, 1] \ni t \mapsto c_t \in \mathbb{C}, \quad \text{such that } \nu_t(c_t) = 0
  \]

  is contained in \( \mathcal{X} \).

- Take \( \mathcal{O}' = \mathcal{O}_2 \) – the unique periodic orbit of period 2.

- \( \rho_{\mathcal{O}_2}(c) = 4c + 4 \), \( \Rightarrow \) \( \nu_1 \) has no zeros in \( \mathbb{C} \) \( \Rightarrow \) the curve \( c_t \) leaves \( \mathcal{X} \setminus \mathcal{M} \), \( \Rightarrow \) connects \( c_0 \) with \( \partial \mathcal{M} \).
\( M \subset X \)

\[ F_k(c) := f_c^{(k-1)}(c). \]

Then \( F_k(c) \) is the free term of the polynomial \( f_c^k(z) \), hence

\[
F_k(c) = 2^{-2^k} \prod_{m \in \mathbb{N}, m | k} \prod_{\mathcal{O} \in \Omega_c^m} \rho_\mathcal{O}(c),
\]

where the product is taken over all \( m \in \mathbb{N} \), such that \( m \) divides \( k \) and over all periodic orbits \( \mathcal{O} \in \Omega_c^m \).

\[
\frac{F'_k(c)}{kF_k(c)} = \sum_{m \in \mathbb{N}, m | k} \sum_{\mathcal{O} \in \Omega_c^m} \frac{m}{k} \nu_\mathcal{O}(c) \to 0,
\]

as \( k \to \infty \) over an appropriate subsequence, provided that \( c \in \text{int}(M) \) is not parabolic or critically periodic.