Accumulation set of critical points of the multipliers in the quadratic family

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Multipliers as functions of the parameter

- Quadratic family: $\{ f_c(z) = z^2 + c \mid c \in \mathbb{C} \}$.
- $\mathcal{O} = \langle z_0, z_1, \ldots, z_{k-1} \rangle$ is a periodic orbit of period k for f_{c_0}.
- Notation: $|\mathcal{O}| := k$ – the period of $\mathcal{O}$.
- Multiplier map: $\rho_{\mathcal{O}}(c) = f_c'(z_0) f_c'(z_1) \ldots f_c'(z_{k-1})$

• $\rho_{\mathcal{O}}$ is a locally analytic function around c_0, whenever $\mathcal{O}$ is not a primitive parabolic orbit of f_{c_0}.

• $\rho_{\mathcal{O}}$ extends to a multiple-valued algebraic function on $\mathbb{C}$.

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Question: What can we say about critical points and critical values of the multiplier maps (i.e., when $\rho_{\mathcal{O}}'(c) = 0$)?

Theorem (Sullivan, Douady-Hubbard): The multiplier $\rho_{\mathcal{O}}$ of an attracting periodic orbit is a Riemann mapping of the corresponding hyperbolic component.

$$\rho_{\mathcal{O}}^{-1} \colon \mathbb{D} \to H \qquad \text{is a conformal isomorphism.}$$

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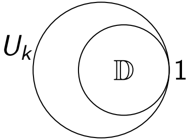
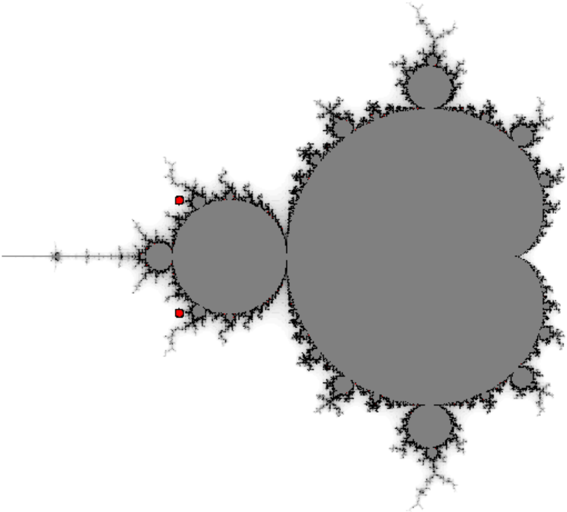
Problem: Does there exist a neighborhood $U \Supset \mathbb{D}$, such that $\rho_{\mathcal{O}}^{-1}$ is univalent in U, for any periodic orbit $\mathcal{O}$?

Theorem (Levin 2009, Dezotti 2011): $\rho_{\mathcal{O}}^{-1}$ is univalent in $U_k \supset \mathbb{D}$, where $\partial U_k \cap \partial \mathbb{D} = \{1\}$, and U_k depends on $k = |\mathcal{O}|$.



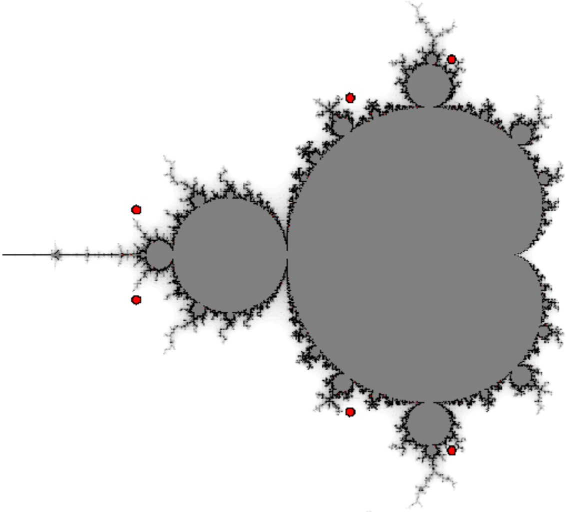
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 3$

Number of critical points = 2



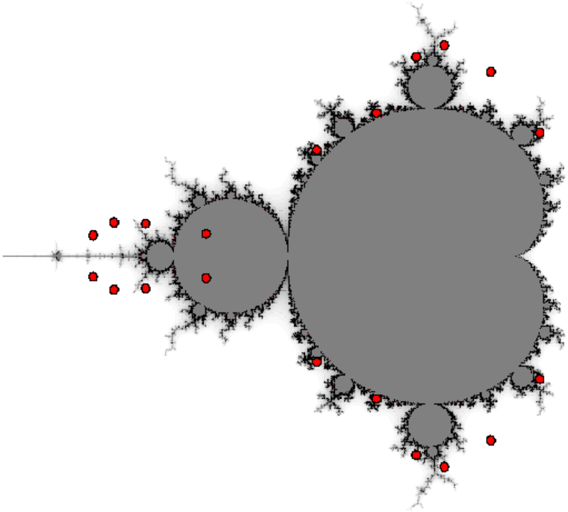
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 4$

Number of critical points = 6



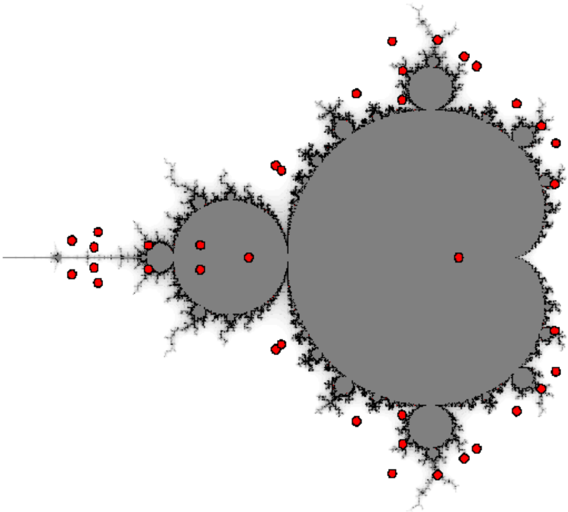
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 5$

Number of critical points $= 20$



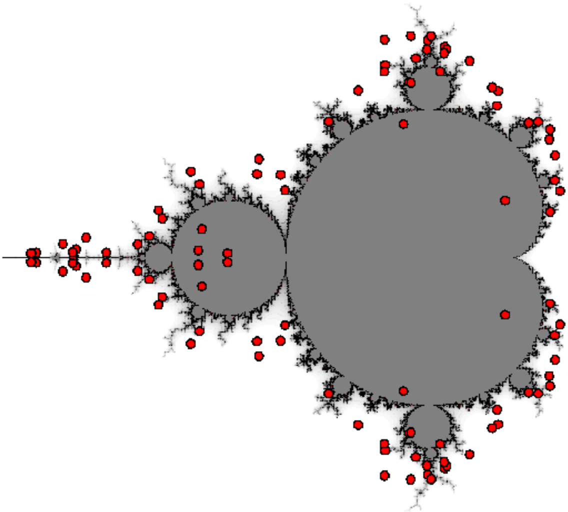
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 6$

Number of critical points = 38



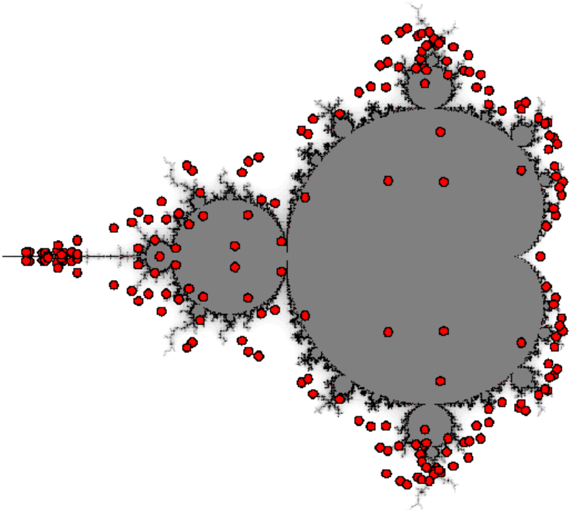
Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 7$

Number of critical points $= 102$

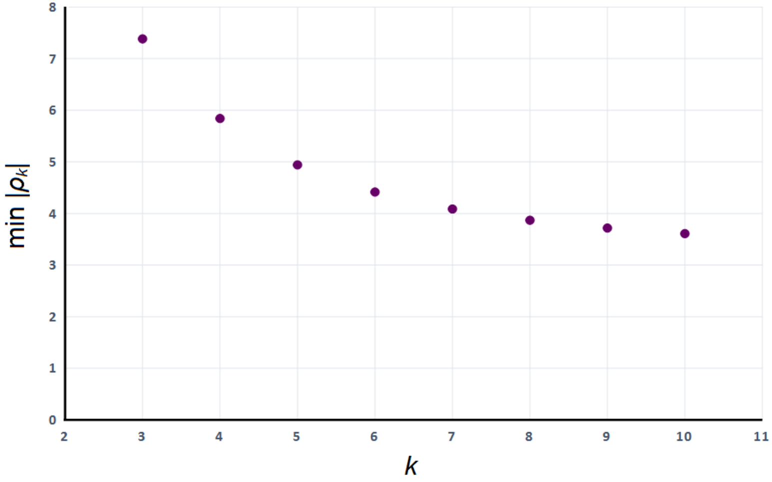


Critical points of the multiplier maps $\rho_{\mathcal{O}}$, $|\mathcal{O}| = 8$

Number of critical points $= 198$



Minimal critical values of the multiplier maps $\rho_{\mathcal{O}}$



Equidistribution of critical points of the multipliers

For any $s \in \mathbb{C}$ and any $k \in \mathbb{N}$,

- $X_{s,k} := \{c \in \mathbb{C} \mid \rho'_{\mathcal{O}}(c) = s, \text{ for some periodic orbit } \mathcal{O}\}$. (Points in $X_{s,k}$ are counted with multiplicity.)

$$\nu_{s,k} := \frac{1}{\#X_{s,k}} \sum_{c \in X_{s,k}} \delta_c.$$

Equidistribution Theorem (Firsova, G. 2019): For every sequence of complex numbers $\{s_k\}_{k\in\mathbb{N}}$, such that

$$\limsup_{k\to+\infty} \frac{1}{k} \log|s_k| \le \log 2,$$

the sequence of measures $\{\nu_{s_k,k}\}_{k\in\mathbb{N}}$ converges to μ_{bif} in the weak sense of measures on $\mathbb{C}$, as $k \to \infty$.

Related results for quadratic polynomials

Theorem (Levin 1989, Bassanelli-Berteloot 2011, Buff-Gauthier 2015): For any $\rho_0 \in \mathbb{C}$, the set of parameters c (counted with multiplicity), such that $\rho_{\mathcal{O}}(c) = \rho_0$, for some $\mathcal{O}$ of period k, equidistributes on the boundary of $\mathbb{M}$, as $k \to \infty$.

Accumulation set: For an infinite collection of points $A \subset \mathbb{C}$ (counted with multiplicities), its accumulation set consists of all points $z \in \mathbb{C}$, whose arbitrary neighborhood contains infinitely many points from A.

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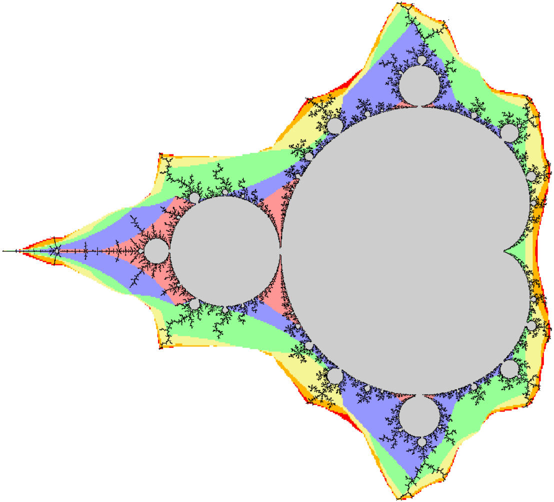
- $\mathcal{X} \subset \mathbb{C}$ is the accumulation set of critical points of the multipliers.

Theorem (Firsova, G. 2020): The accumulation set $\mathcal{X}$ is bounded, path connected and contains the Mandelbrot set $\mathbb{M}$. Furthermore, the set $\mathcal{X} \setminus \mathbb{M}$ is nonempty and has a nonempty interior, and every critical point of any multiplier is in $\mathcal{X}$.

The accumulation set $\mathcal{X}$



Roots of the multipliers and Lyapunov exponents

The root of the multiplier of a periodic orbit $\mathcal{O}$:

$$g_{\mathcal{O}}(c) := [\rho_{\mathcal{O}}(c)]^{1/|\mathcal{O}|}$$

The Lyapunov exponent of an arbitrary orbit $z_0, z_1, z_2, \ldots$:

$$\lambda_c(z_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f_c'(z_j)|$$

For a periodic orbit $\mathcal{O} = \langle z_0, \ldots, z_{k-1} \rangle$,

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Ergodic Theorem: There exists a function $L : \mathbb{C} \to \mathbb{R}$, such that $\lambda_c(z_0) = L(c)$, for a.e. $z_0 \in J_c$ with respect to the harmonic measure on J_c.

Przytycki's formula: $L(c) = \log 2 + \frac{1}{2} G_{\mathbb{M}}(c)$.

Roots of the multiplier maps in $\mathbb{C} \setminus \mathbb{M}$

- Ω_c^k is the set of all period k cycles of f_c, for $c \in \mathbb{C}$.

Lemma: For any $\delta > 0$ and a compact subset $K \subset \mathbb{C} \setminus \mathbb{M}$, the following holds:

$$\lim_{k\to\infty} \frac{\#\{\mathcal{O} \in \Omega_c^k \colon \| \log |g_{\mathcal{O}}| - L\|_K < \delta\}}{\#\Omega_c^k} = 1$$

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or equivalently,

$$\lim_{k\to\infty} \frac{\#\{\mathcal{O} \in \Omega_c^k \colon \|g_{\mathcal{O}} - 2\sqrt{\phi_{\mathbb{M}}}\|_K < \delta\}}{\#\Omega_c^k} = 1,$$

where

$$\phi_{\mathbb{M}} \colon \mathbb{C} \setminus \mathbb{M} \to \mathbb{C} \setminus \overline{\mathbb{D}} \quad \text{– conformal diffeomorphism.}$$

The sets $\mathcal{Y}_c$

- Ω_c is the set of all repelling periodic orbits of f_c.
- For every $\mathcal{O} \in \Omega_{c_0}$, the function

$$\nu_{\mathcal{O}}(c) := \frac{\rho'_{\mathcal{O}}(c)}{|\mathcal{O}|\,\rho_{\mathcal{O}}(c)} = [\log g_{\mathcal{O}}(c)]'$$

  is defined and analytic around $c = c_0$.
- For each $c \in \mathbb{C}$, we consider the set $\mathcal{Y}_c \subset \mathbb{C}$, defined by

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Theorem (Firsova, G. 2020): The following two properties hold:

(i) For every parameter $c \in \mathbb{C} \setminus \{-2\}$, the set $\mathcal{Y}_c$ is convex; for $c = -2$, the set $\mathcal{Y}_{-2}$ is the union of a convex set and the point $-\frac{1}{6}$.

(ii) For every parameter $c \in \mathbb{C} \setminus \mathbb{M}$, the set $\mathcal{Y}_c$ is bounded. A parameter $c \in \mathbb{C} \setminus \mathbb{M}$ belongs to $\mathcal{X}$, if and only if $0 \in \mathcal{Y}_c$.

Critical points of the Hausdorff dimension function

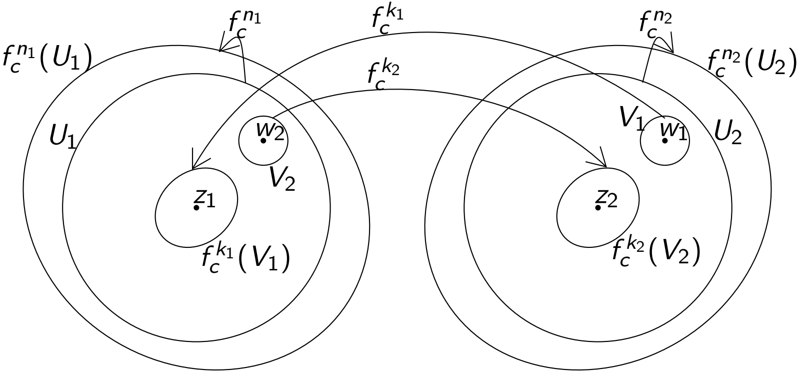
Hausdorff dimension function: $\delta(c) := \dim_H(J_c)$

Theorem (Bowen): The function δ is real-analytic in each hyperbolic component (including the complement of $\mathbb{M}$).

Theorem (Y. M. He, H. Nie 2020): *(Version for the quadratic family)* If $c \in \mathbb{C}$ is a hyperbolic parameter and $0 \notin \mathcal{Y}_c$, then c is not a critical point of the function δ.

Corollary: The Hausdorff dimension function δ has no critical points in $\mathbb{C} \setminus \mathcal{X}$.

Proof of (i): Averaging Lemma



Averaging Lemma: Let $\mathcal{O}_1, \mathcal{O}_2$ be two distinct non-exceptional repelling periodic orbits of f_c. Then for any $t \in [0, 1]$, there exists a sequence of periodic orbits $\mathcal{O}_3, \mathcal{O}_4, \dots$ of f_c, such that

$$g_{\mathcal{O}_j} \to g_{\mathcal{O}_1}^t g_{\mathcal{O}_2}^{1-t}, \quad \text{and} \quad \nu_{\mathcal{O}_j} \to t\nu_{\mathcal{O}_1} + (1-t)\nu_{\mathcal{O}_2}$$

uniformly on a neighborhood of c for appropriate branches of the powers.

Proof of (ii)

$$\nu_{\mathcal{O}}(c) := \frac{\rho'_{\mathcal{O}}(c)}{|\mathcal{O}|\,\rho_{\mathcal{O}}(c)} = [\log g_{\mathcal{O}}(c)]'$$

Lemma: Let $U \subset \mathbb{C} \setminus \partial\mathbb{M}$ be an open domain and fix $c \in U$. Then each map from the family

$$\mathcal{F}_c = \{\nu_{\mathcal{O}} \mid \mathcal{O} \in \Omega_c\}$$

is defined in U, and $\mathcal{F}_c$ is normal in U.

Corollary: For every $c \in \mathbb{C} \setminus \partial\mathbb{M}$, we have

$$\mathcal{Y}_c = \{\nu(c) \mid \nu \in \overline{\mathcal{F}_c}\}.$$

Proof of (ii): $c \in \mathcal{X} \iff \exists$ a sequence of points $c_j \to c$ and a sequence of orbits $\mathcal{O}_j \in \Omega_c$, such that $\nu_{\mathcal{O}_j}(c_j) = 0$, $\implies \exists$ a map $\nu \in \overline{\mathcal{F}_c}$, such that $\nu(c) = 0 \iff 0 \in \mathcal{Y}_c$.

" $\Longleftarrow$ " also holds if $\nu \not\equiv 0$.

Lemma: If $c \in \mathbb{C} \setminus \mathbb{M}$, then $0 \notin \overline{\mathcal{F}_c}$.

$\mathcal{X}$ is bounded and path connected

Lemma: The set $\mathcal{X}$ is bounded.

Idea of the proof: Normality of the family $\{g_{\mathcal{O}} \mid \mathcal{O} \in \Omega_c\}$ in $\mathbb{C} \setminus \mathbb{M}$.

Lemma: The set $\mathcal{X} \cup \mathbb{M}$ is path connected.

Idea of the proof:

- Let $c_0 \in \mathcal{X} \setminus \mathbb{M}$ and $\nu_{\mathcal{O}}(c_0) = 0$, for some orbit $\mathcal{O}$. Let $\mathcal{O}'$ be another orbit, and consider

$$\nu_t := (1-t)\nu_{\mathcal{O}} + t\nu_{\mathcal{O}'}, \quad \text{for } t \in [0,1].$$

Then the curve

$$[0,1] \ni t \mapsto c_t \in \mathbb{C}, \quad \text{such that } \nu_t(c_t) = 0$$

is contained in $\mathcal{X}$.

- Take $\mathcal{O}' = \mathcal{O}_2$ – the unique periodic orbit of period 2.
- $\rho_{\mathcal{O}_2}(c) = 4c + 4$, $\implies$ ν_1 has no zeros in $\mathbb{C}$ $\implies$ the curve c_t leaves $\mathcal{X} \setminus \mathbb{M}$, $\implies$ connects c_0 with ∂M.

$\mathbb{M} \subset \mathcal{X}$

- $F_k(c) := f_c^{\circ(k-1)}(c)$.

Then $F_k(c)$ is the free term of the polynomial $f_c^{\circ k}(z)$, hence

$$F_k(c) = 2^{-2^k} \prod_{m\in\mathbb{N}, m|k} \prod_{\mathcal{O}\in\Omega_c^m} \rho_{\mathcal{O}}(c),$$

where the product is taken over all $m \in \mathbb{N}$, such that m divides k and over all periodic orbits $\mathcal{O} \in \Omega_c^m$.

$$\frac{F_k'(c)}{kF_k(c)} = \sum_{m\in\mathbb{N}, m|k} \sum_{\mathcal{O}\in\Omega_c^m} \frac{m}{k}\nu_{\mathcal{O}}(c) \to 0,$$

as $k \to \infty$ over an appropriate subsequence, provided that $c \in \mathrm{int}(\mathbb{M})$ is not parabolic or critically periodic.