Julia sets in non-autonomus quadratic iteration

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Setting and notation

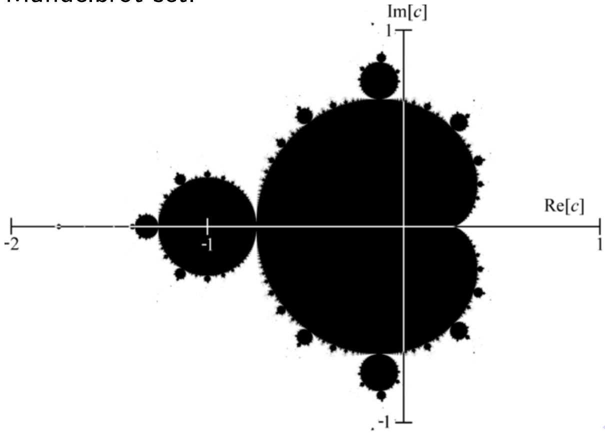
- We will consider compositions of functions $f_{c_n}(z) = z^2 + c_n$.
- Let $\omega = (c_0, c_1, c_2, ...) \in \mathbb{C}^{\mathbb{N}}$ be a sequence of complex numbers.
- We denote $f_\omega^n(z) = f_{c_{n-1}}(z) \circ f_{c_{n-2}}(z) \circ ... \circ f_{c_0}(z)$.
- For a given sequence ω we can ask questions about normality of the family $\{f_\omega^n\}$.

Non-autonomous definitions

- The Fatou set is defined by
  $\mathbb{F}(\omega) = \{z \in \widehat{\mathbb{C}} : \{f_\omega^n\}$ *is normal on a neighbourhood of* z $\}$
- The Julia set $\mathbb{J}(\omega)$ is the complement of the Fatou set.
- In the autonomous case these sets depend on a parameter c, since we investigate the normality of iterations of $z^2 + c$. In our case these sets depend on a sequence ω.

Autonomous quadratic iteration

- The autonomous case where $\forall_n c_n = c$ has been studied extensively.
- The Julia set $\mathbb{J}_c$ is in this case either connected, or totally disconnected, i.e. every connected component is a single point.
- The set of points c for which the Julia set is connected is the famous Mandelbrot set.



Trajectory of 0

Let $\sigma : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the left shift map, so that $\sigma((c_0, c_1, \ldots)) = (c_1, c_2, \ldots)$.

Theorem

The Julia set for $f_c(z) = z^2 + c$ is disconnected if and only if $f_c^n(0) \xrightarrow[n\to\infty]{} \infty$.

Theorem (R. Brück, M. Büger, S. Reitz, 1999)

The Julia set $\mathbb{J}_\omega$ is disconnected, if and only if there exists $k \in \mathbb{N}$ such that $f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty$.

Note that in the non-autonomous case the Julia set doesn't have to be totally disconnected. For instance take $c_0 = 10$ and $c_n = 0$ for $n > 0$, the Julia set in this case has 2 connected components, the preimages of the unit circle by $z^2 + 10$.

Total disconnectedness of the Julia set

Theorem (R. Brück, M. Büger, S. Reitz, 1999)

There exists a sequence ω such that $f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty$ for all $k \in \mathbb{N}$ but the Julia set is not totally disconnected.

Theorem (R. Brück, M. Büger, S. Reitz, 1999)

There exists a sequence ω such that the Julia set J_ω is totally disconnected but $f^n_{\sigma^k\omega}(0)$ does not converge to infinity for infinitely many k.

Typical Julia set

Remark

If $R \leqslant \frac{1}{4}$ then for every $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$ the Julia set $\mathbb{J}_\omega$ is connected.

What typically happens when we choose $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$ if $R > \frac{1}{4}$?

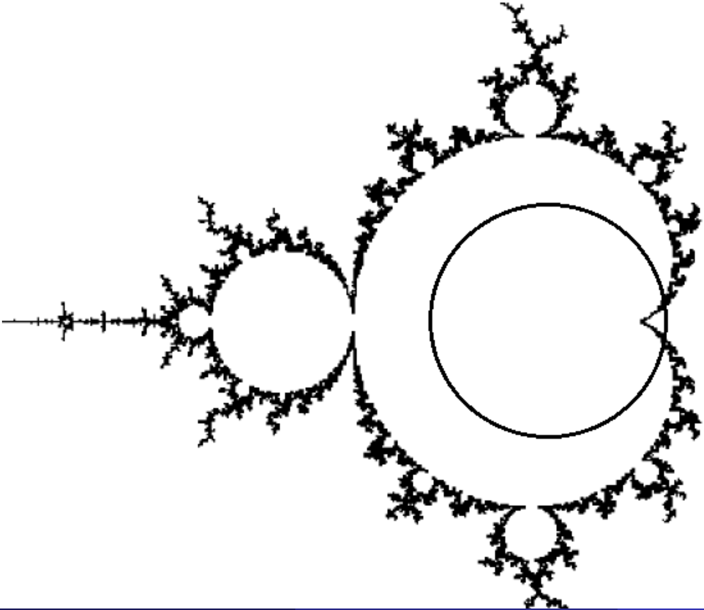
Theorem (Z. Gong, W. Qiu, Y. Li, 2003)

Let $R > \frac{1}{4}$, then the set of $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$ for which the Julia set $\mathbb{J}_\omega$ is totally disconnected is of second Baire category.

Question

If $R > \frac{1}{4}$ is the Julia set typically (in the sense of product of Lebesgue measures) totally disconnected?

Typical Julia set



Typical Julia set

Let λ_R be the Lebegue measure on $\mathbb{D}(0, R)$ normalized so that $\lambda_R(\mathbb{C}) = 1$. We denote by $\mathbb{P}$ the product measure $\otimes_{n=0}^{\infty}\lambda_R$ defined on $\mathbb{D}(0, R)^{\mathbb{N}}$.

Theorem (L., A. Zdunik, 2020)

Let $R > \frac{1}{4}$, then for $\mathbb{P}$-almost every sequence $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$ the Julia set $\mathbb{J}_\omega$ is totally disconnected.

Theorem (L., A. Zdunik, 2020)

Let V be an open and bounded set such that $\mathbb{D}(0, \frac{1}{4}) \subset V$ and $V \neq \mathbb{D}(0, \frac{1}{4})$. Consider the space $\Omega = V^{\mathbb{N}}$ equipped with the product $\mathbb{P}$ of uniform distributions on V. Then for $\mathbb{P}$-almost every sequence $\omega \in \Omega$ the Julia set $\mathbb{J}_\omega$ is is totally disconnected.

General idea of the proof

Remark

If a disk D centered at 0 has large enough radius, then the filled -in Julia set K_ω, i.e. the set of points z whose trajectories $f_\omega^n(z)$ do not escape to ∞ can be written as

$$K_\omega := \bigcap_{k\in\mathbb{N}} (f_\omega^k)^{-1}(D),$$

for any $\omega \in D(0,R)^{\mathbb{N}}$.

Lemma

Let $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$. If there exists $N \in \mathbb{N}$ such that for infinitely many indices $k \in \mathbb{N}$, for each component $D_j^k(\omega)$ of the set $D^k(\omega) = (f_\omega^k)^{-1}(D)$ the degree of the map

$$f_\omega^k : D_j^k(\omega) \to D$$

is at most N, then the Julia set J_ω is totally disconnected.

General idea of the proof

Lemma

If there exists a γ such that the following holds

$$\mathbb{P}\left(\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\}\right) < e^{-\gamma k}, \tag{1}$$

where g_ω is the green function for the basin of infinity of f_ω^n and G is a constant dependent on R, then the assumptions of the previous lemma are satisfied.

Lemma

Let V be a bounded open set such that $D(0, \frac{1}{4}) \subset V$ and $V \neq D(0, \frac{1}{4})$. Take $\Omega = V^{\mathbb{N}}$ to be the product space equipped with the product of uniform distributions on V, denoted by $\mathbb{P}$. There exists a constant $\gamma > 0$ such that

$$\mathbb{P}\left(\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\}\right) < e^{-\gamma k}.$$