Mating quadratic maps with the modular group

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Matings

- A mating between 2 objects A and B is an object C behaving as A on an invariant subset of its domain, and as B on the complement.

- They exist in both worlds of rational maps and of Kleinian groups.

- Both rat. & Klein. when iterated on \( \hat{C} \) divide \( \hat{C} \) in 2 invariant sets:
  - domain of normality (Fatou set, ordinary set resp),
  - the complement (Julia set, limit set resp).

- Can we mate a rational map (on an invariant component of its Fatou set) and a Kleinian group (on an invariant component of its ordinary) set?
  1. Do rational maps and kleinian groups fit together? In some object C?
  2. Is C a mating? Is a family of C a family of matings?
Bullett-Penrose: $PSL(2, \mathbb{Z})$ and quadratic maps fit together

Modular group $PSL(2, \mathbb{Z})$:
Kleinian group with generators:

$$\tau_1(z) = z + 1, \text{ and } \tau_2(z) = \frac{z}{1 + z}$$

Minkowski map $h_+ : [0, \infty) \rightarrow [0, 1]$: homeo

$$x \in \mathbb{R}, \text{ written } [x_0; x_1, x_2, \ldots] = x_0 + \frac{1}{x_1 + \frac{1}{x_2 + \ldots}}$$

$$h_+(x) = 0.1\ldots10\ldots01\ldots1\ldots$$

It conjugates the action of $\tau_1(z)$ and $\tau_2(z)$ on $(-\infty, 0]$ with the doubling map, and on $[0, \infty)$ with the halving map.
Holomorphic correspondences

A 2 : 2 holomorphic correspondence $\mathcal{F}$ on $\hat{\mathbb{C}}$: is a multi-valued map $\mathcal{F} : z \to w$ defined by a polynomial relation $P(z, w) = 0$ of deg 2 in $z$ and 2 in $w$.

Rational map $f$, deg$(f)$=2, $\longleftrightarrow$ (2:1) $\mathcal{F} : z \to w$

$f(z) = \frac{p(z)}{q(z)}$

with $P(z, w) = wq(z) − p(z)$

Modular group $\longleftrightarrow$ (2:2) $\mathcal{F} : z \to w$

with generators

$\tau_1(z) = z + 1$, $\tau_2(z) = \frac{z}{1 + z}$
Quadratic maps and $PSL(2, \mathbb{Z})$ fit in a correspondence

Theorem (Bullett-Penrose, ’94) The matings between $Q_c, c \in M$ and $PSL(2, \mathbb{Z})$ lie in the family $\mathcal{F}_a : z \rightarrow w$ given by

$$
\left( \frac{aw - 1}{w - 1} \right)^2 + \left( \frac{aw - 1}{w - 1} \right) \left( \frac{az + 1}{z + 1} \right) + \left( \frac{az + 1}{z + 1} \right)^2 = 3.
$$

Moreover, for all $a \in [4, 7] \subset \mathbb{R}$ the correspondence $\mathcal{F}_a$ is conjugate to the generators of $PSL(2, \mathbb{Z})$ on an invariant open set.

**Figure:** Limit set of $\mathcal{F}_a, a = (3 + \sqrt{33})/2$
$M_\Gamma$ and some limit sets for $\mathcal{F}_a$
Which correspondences in $\mathcal{F}_a$ are matings? 
Conjecture and state of the art

◮ **Conjecture (B-P, ’94)** The family $\mathcal{F}_a$ contains matings between $PSL(2, \mathbb{Z})$ and every quadratic polynomial with connected Julia set, and the connectedness locus $M_\Gamma$ of $\mathcal{F}_a$ is homeomorphic to $M$.

The family $F_a = J_a \circ Cov_Q^0$

$Q(z) = z^3 - 3z$, $Cov_Q^0$ is its correspondence 'deck transformation':
$Q(x) = Q(y) = Q(z) \Rightarrow Cov_Q^0(x) = \{y, z\}$
$J_a$ involution having fixed points at 1 (critical point of $Q$) and $a$.

$Q(-2) = Q(1)$, so $Cov_Q^0$ sends each blue line to the other two.

$\Delta Cov_Q^0$ fundamental domain for $Cov_Q^0$
Dynamics of the family $\mathcal{F}_a$

Facts: for every $a$

- $\mathcal{F}_a$ has a parabolic fixed point at $z = 0$.
- $\mathcal{F}_a^{-1}(0) = \{0, S_a\}$.
- $\mathcal{F}_a|_{\Delta_a}$ is a deg 2 holomorphic map at every $z \in \Delta_a$, $z \neq S_a$. 

$\Lambda_{a,-} = \bigcap_{i=1}^{\infty} (\mathcal{F}_a)^{-i}(\Delta_a)$

$\Lambda_{a,+} = \bigcap_{i=1}^{\infty} (\mathcal{F}_a)^{i}(\bar{\mathbb{C}} \setminus \Delta_a)$
The family $\text{Per}_1(1)$

- $\text{Per}_1(1) = \{P_A(z) = z + 1/z + A \mid A \in \mathbb{C}\}$,
- $\infty$ parabolic fixed point, with basin $\Lambda_A$, $K_A = \hat{\mathbb{C}} \setminus \overline{\Lambda}_A$
- $\forall A$, external class given by $h_2(z) = \frac{z^2 + 1/3}{1 + z^2/3}$, and $h_2(z)|_{S^1} \sim_{\text{top}} P_0(z) = z^2|_{S^1}$
- $M_1$: connectedness locus ($M_1 \approx M$ by Petersen-Roesch).
Main Theorem

We say that $F_a$ is a mating between the rational quadratic map $P_A : z \to z + 1/z + A$ and the modular group $\Gamma = PSL(2, \mathbb{Z})$ if

1. the 2-to-1 branch of $F_a$ for which $\Lambda_-$ is invariant, is hybrid equivalent to $P_A$ on $\Lambda_-$ (i.e., it is quasiconformally conjugate to $P_A$ in a nbh of $\Lambda_-$ by a map which is conformal on the interior of $\Lambda_-$),

2. when restricted to a $(2 : 2)$ correspondence from $\Omega(F_a)$ to itself, $F_a$ is conformally conjugate to the pair of Möbius transformations $\{\tau_1, \tau_2\}$ from the complex upper half plane $\mathbb{H}$ to itself.


For every $a \in M_\Gamma$ the correspondence $F_a$ is a mating between some rational map $P_A : z \to z + 1/z + A$ and $\Gamma$. 
$M_1$ and $M_Γ$
A (deg 2) parabolic-like map is an object that locally behaves like a map $P_A, A \in \mathbb{C}$.

**Thm** A deg 2 parabolic-like map is hybrid conjugated to a member of the family $\text{Per}_1(1)$, a unique such member if the filled Julia set is connected.

So, if we prove that for all $a \in M_\Gamma$, the branch of $\mathcal{F}_a$ which fixes $\Lambda_-$ restricts to a parabolic-like map, we are done!
Parabolic-like maps
Parabolic-like maps
Parabolic-like maps
Parabolic-like maps
From correspondences to parabolic-like maps
From correspondences to parabolic-like maps
From correspondences to parabolic-like maps: qc surgery to kill one image
Surgery construction (valid also out of $M_\Gamma$)

$\Lambda - \subset \mathbb{H}_I$, but we need it to be contained in a top. disc making at the parabolic fixed point an angle $< \pi$, to have space to put our new Beltrami forms!

'Easy' to do individually
Part 2: Böttcher map (valid just in $M_{\Gamma}$)

- $\forall a \in M_{\Gamma} \exists$ a Riemann map $\phi : \Omega \to \mathbb{H}$. We prove that $\phi$ conjugates $\mathcal{F}_a|_{\Omega(\mathcal{F}_a)}$ to the generators of the modular groups on $\mathbb{H}$.

- This is: the map $\phi : \Omega \to \mathbb{H}$ plays for our family $\mathcal{F}_a$ the role the Böttcher map plays for the quadratic family!
Proving $M_\Gamma \approx M_1$

The straightening construction induces a map

$$\chi : M_\Gamma \to M_1$$

The map $\chi$ is injective (punch-line: Rickmann Lemma).

**Theorem (Bullett-L)** The map $\chi : M_\Gamma \setminus \{4, 7\} \to M_1 \setminus \{-3, 1\}$ is a homeomorphism, which extends to a doubly pinched neighbourhood of $M_\Gamma$. 
Main tool: lunes and fundamental croissants

$M_{\Gamma} \subset \text{lune } L \Rightarrow \text{for all } a \in M_{\Gamma}, \Lambda_{-,a} \text{ is contained in a lune } V_a \text{ which moves holomorphically with the parameter.}$

For $a \in L$, set $V'_a := \mathcal{F}_a^{-1}(V_a)$. We call $\mathcal{A}_a := V_a \setminus V'_a$ the fundamental croissant.

The fundamental croissant moves holomorphically for $a \in \hat{L}$: fix $a_0 \in M_{\Gamma} \setminus \{4, 7\}$, we have a holomorphic motion

$$\tau_a : \mathcal{A}_{a_0} \rightarrow \mathcal{A}_a$$

We can extend $\chi : \hat{L} \setminus M_{\Gamma} \rightarrow \mathbb{C}$ by using $\tau$: the extension is qc.
Holomorphic motion of Beltrami forms

Do surgery at $a_0 \in M_\Gamma$, obtaining $\mu_{a_0}$ on $\mathcal{A}_{a_0}$, move it by $\tau : \mathcal{A}_{a_0} \to \mathcal{A}_a$ $\mu_a = (\tau_a)_*(\mu_{a_0})$, and $\overline{\mu}_a = (\mathcal{F}_a)^{-n}(\mu_a)$ on $\mathcal{F}_a^{-n}(\mathcal{A}_a)$ for all $n \geq 1 \Rightarrow \overline{\mu}_a$ is holomorphic in $a$ wherever $\Lambda_{a,-}$ moves holomorphically with $a$.

Technical: we can construct $\gamma_a$ holomorphic with $a$ for $a$ in a doubly pinched neighbourhood $U(M_\Gamma)$ of $M_\Gamma$ (pinched at 4 and 7)
Last steps

- $U(M_{\Gamma}) \setminus \partial M_{\Gamma}$ is the set of parameters where $\Lambda_{a,-}$ moves holomorphically (by MSS decomposition).

- $\chi$ holomorphic on $\tilde{M}_{\Gamma}$ by holomorphic motion arguments

- + qc. out of $M_{\Gamma} \Rightarrow \chi$ continuous on $U(M_{\Gamma}) \setminus \partial M_{\Gamma}$

- sequences of qc maps have convergent subs. + rigidity on $\partial M_1 \Rightarrow \chi$ continuous on $\partial M_{\Gamma}$.

- $\chi$ is a branched covering, and as it is injective on $M_{\Gamma}$, it is a homeo on $U(M_{\Gamma})$. 
### Dictionary between $Q_c$ and $\mathcal{F}_a$

<table>
<thead>
<tr>
<th>Quadratic polynomials $Q_c$</th>
<th>Quadratic correspondences $\mathcal{F}_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_c : \hat{\mathbb{C}} \setminus K(Q_c) \cong \hat{\mathbb{C}} \setminus \overline{D}$</td>
<td>$\varphi_a : \hat{\mathbb{C}} \setminus \Lambda(\mathcal{F}_a) \cong \mathbb{H}$</td>
</tr>
<tr>
<td>external rays</td>
<td>external geodesics</td>
</tr>
<tr>
<td>‘periodic rays land’</td>
<td>‘periodic geodesics land’</td>
</tr>
<tr>
<td>every repelling fixed point</td>
<td>every repelling fixed point</td>
</tr>
<tr>
<td>is the landing point of a periodic ray</td>
<td>is the landing point of a periodic geodesic</td>
</tr>
<tr>
<td>Yoccoz inequality</td>
<td>Yoccoz inequality</td>
</tr>
<tr>
<td>$\sim 1/q$</td>
<td>$\sim (\log q)/q^2$</td>
</tr>
<tr>
<td>$M \subset \overline{D(0, 2)}$</td>
<td>$M_\Gamma \subset \mathcal{L}_\theta$</td>
</tr>
</tbody>
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- External rays and external geodesics correspond to the boundary of the Mandelbrot set and the parameter plane, respectively.
- Periodic rays landing at every repelling fixed point are analogous to periodic geodesics landing at every repelling fixed point.
- Yoccoz inequality relates the Hausdorff dimension of the parameter space to the external rays.

![Mandelbrot set](image1)

![Parameter plane](image2)
Thanks for your attention!