Mating quadratic maps with the modular group

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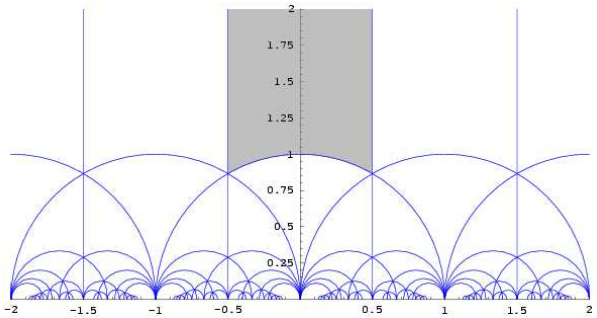
Matings

- A mating between 2 objects A and B is an object C behaving as A on an invariant subset of its domain, and as B on the complement
- They exist in both worlds of rational maps and of Kleinian groups.
- Both rat. & Klein. when iterated on $\widehat{\mathbb{C}}$ divide $\widehat{\mathbb{C}}$ in 2 invariant sets:
  - domain of normality (Fatou set, ordinary set resp),
  - the complement (Julia set, limit set resp).
- Can we mate a rational map (on an invariant component of its Fatou set) and a Kleinian group (on an invariant component of its ordinary) set?
  1. Do rational maps and kleinian groups fit together? In some object C?
  2. Is C a mating? Is a family of C a family of matings?

Bullett-Penrose: $PSL(2,\mathbb{Z})$ and quadratic maps fit together

Modular group $PSL(2,\mathbb{Z})$:
Kleinian group with generators:

$$\tau_1(z) = z + 1, \text{ and } \tau_2(z) = \frac{z}{1+z}$$



Minkowski map $h_+ : [0,\infty) \to [0,1]$: homeo

$$x \in \mathbb{R}, \text{ written } [x_0; x_1, x_2, \ldots] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \ldots}}$$

$$h_+(x) = 0.\underbrace{1\ldots1}_{x_0}\underbrace{0\ldots0}_{x_1}\underbrace{1\ldots1}_{x_2}\ldots$$

It conjugates the action of $\tau_1(z)$ and $\tau_2(z)$ on $(-\infty, 0]$ with the doubling map, and on $[0,\infty)$ with the halving map.

Holomorphic correspondences

A $2:2$ **holomorphic correspondence** $\mathcal{F}$ on $\widehat{\mathbb{C}}$: is a multi-valued map $\mathcal{F}: z \to w$ defined by a polynomial relation $P(z,w)=0$ of deg 2 in z and 2 in w.

Rational map f, deg(f)=2, $f(z)=\frac{p(z)}{q(z)}$	$\longleftrightarrow$	(2:1) $\mathcal{F}: z \to w$ with $P(z,w)=wq(z)-p(z)$
Modular group with generators $\tau_1(z)=z+1,\ \tau_2(z)=\frac{z}{1+z}$	$\longleftrightarrow$	(2:2) $\mathcal{F}: z \to w$ $P(z,w)=(w-(z+1))(w(z+1)-z)$

Quadratic maps and $PSL(2,\mathbb{Z})$ fit in a correspondence

Theorem(Bullett-Penrose, '94) The matings between $Q_c,\ c \in M$ and $PSL(2,\mathbb{Z})$ lie in the family $\mathcal{F}_a : z \to w$ given by

$$\left(\frac{aw-1}{w-1}\right)^2 + \left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 = 3.$$

Moreover, for all $a \in [4,7] \subset \mathbb{R}$ the correspondence $\mathcal{F}_a$ is conjugate to the generators of $PSL(2,\mathbb{Z})$ on an invariant open set.

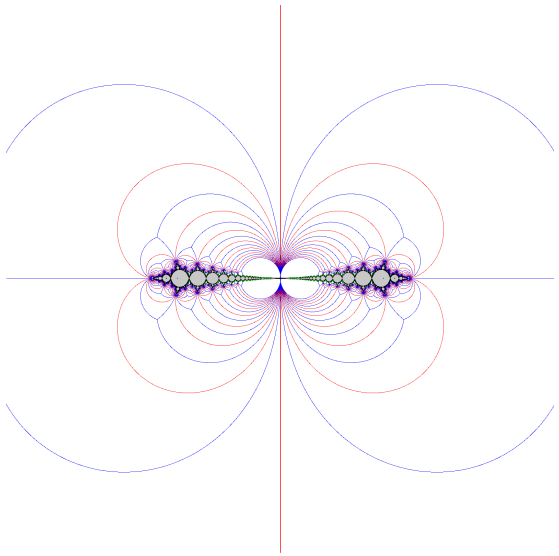
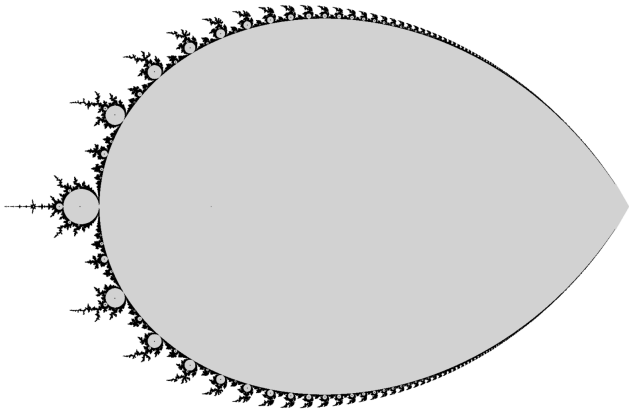
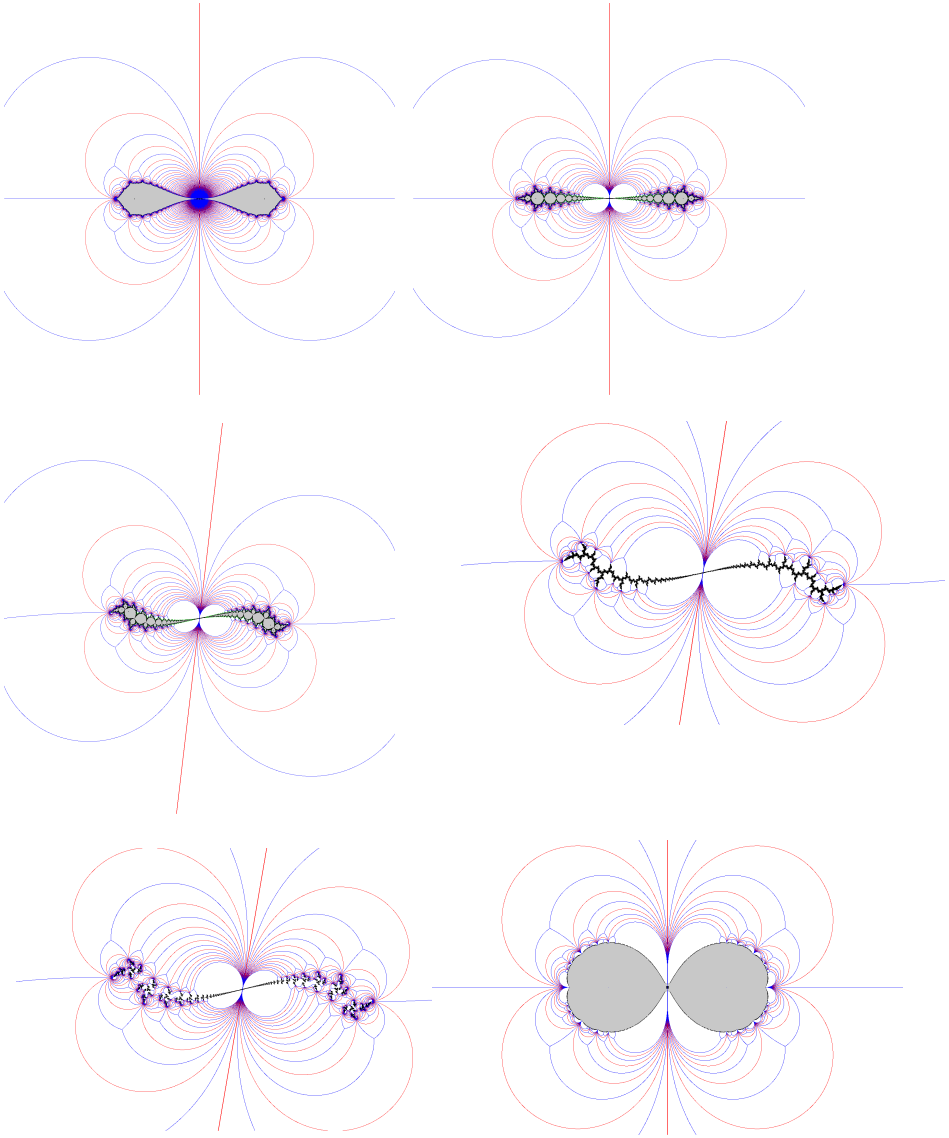


Figure: Limit set of $\mathcal{F}_a$, $a = (3+\sqrt{33})/2$

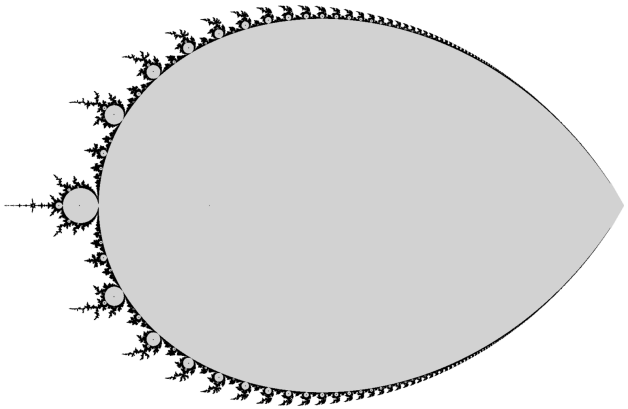
M_Γ and some limit sets for $\mathcal{F}_a$

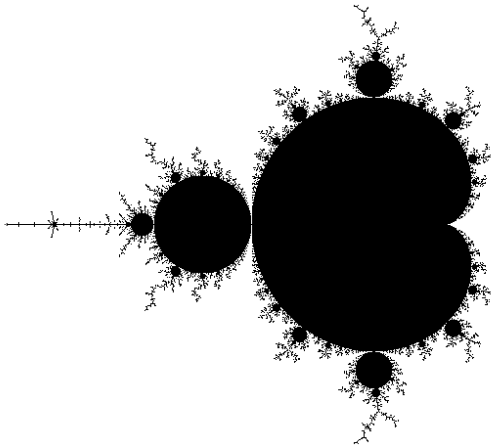




Which correspondences in $\mathcal{F}_a$ are matings? Conjecture and state of the art

- **Conjecture (B-P, '94)** The family $\mathcal{F}_a$ contains matings between $PSL(2,\mathbb{Z})$ and every quadratic polynomial with connected Julia set, and the connectedness locus M_Γ of $\mathcal{F}_a$ is homeomorphic to M.



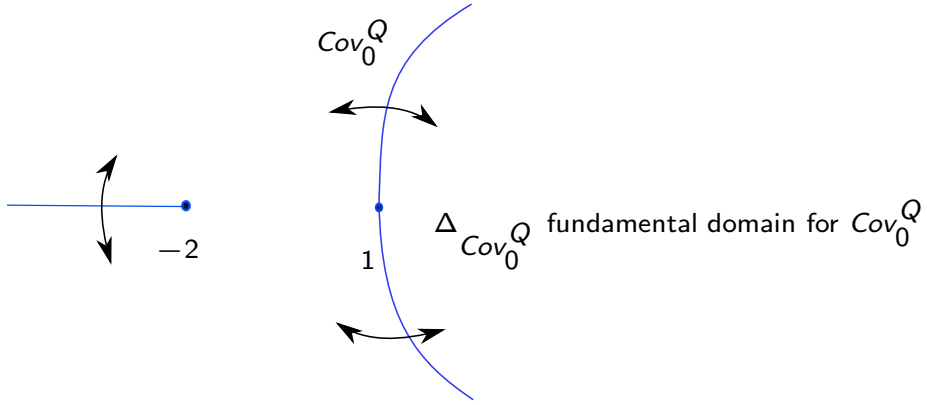


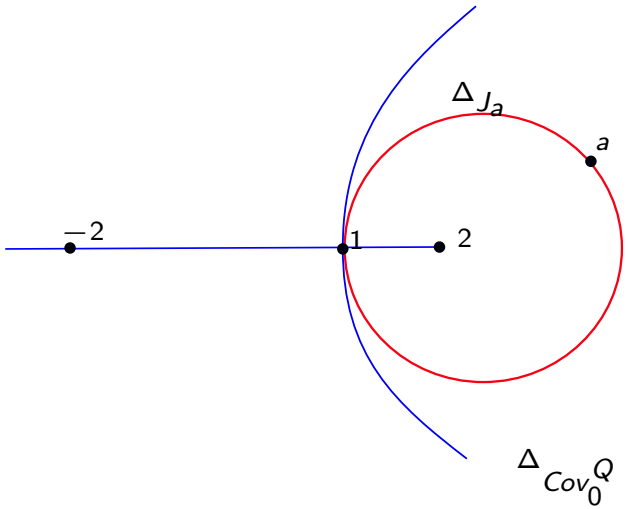
- **Theorem** (B-P,'94 + B-Harvey,*Electron.Res.Announc.AMS*,'00 + B-Haissinsky, *Conform.Geom.Dyn.*,'07) There exists a mating between $PSL(2,\mathbb{Z})$ and Q_c for a large class of value of c

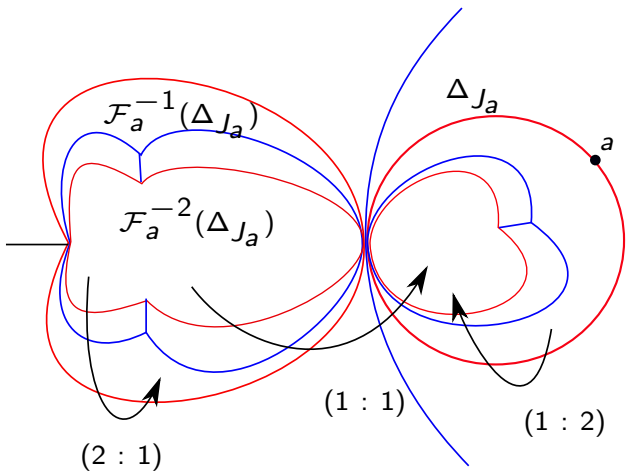
The family $F_a = J_a \circ Cov_Q^0$

$Q(z) = z^3 - 3z$, Cov_Q^0 is its correspondence 'deck transformation':
$Q(x) = Q(y) = Q(z) \Rightarrow Cov_Q^0(x) = \{y, z\}$
J_a involution having fixed points at 1 (critical point of Q) and a.

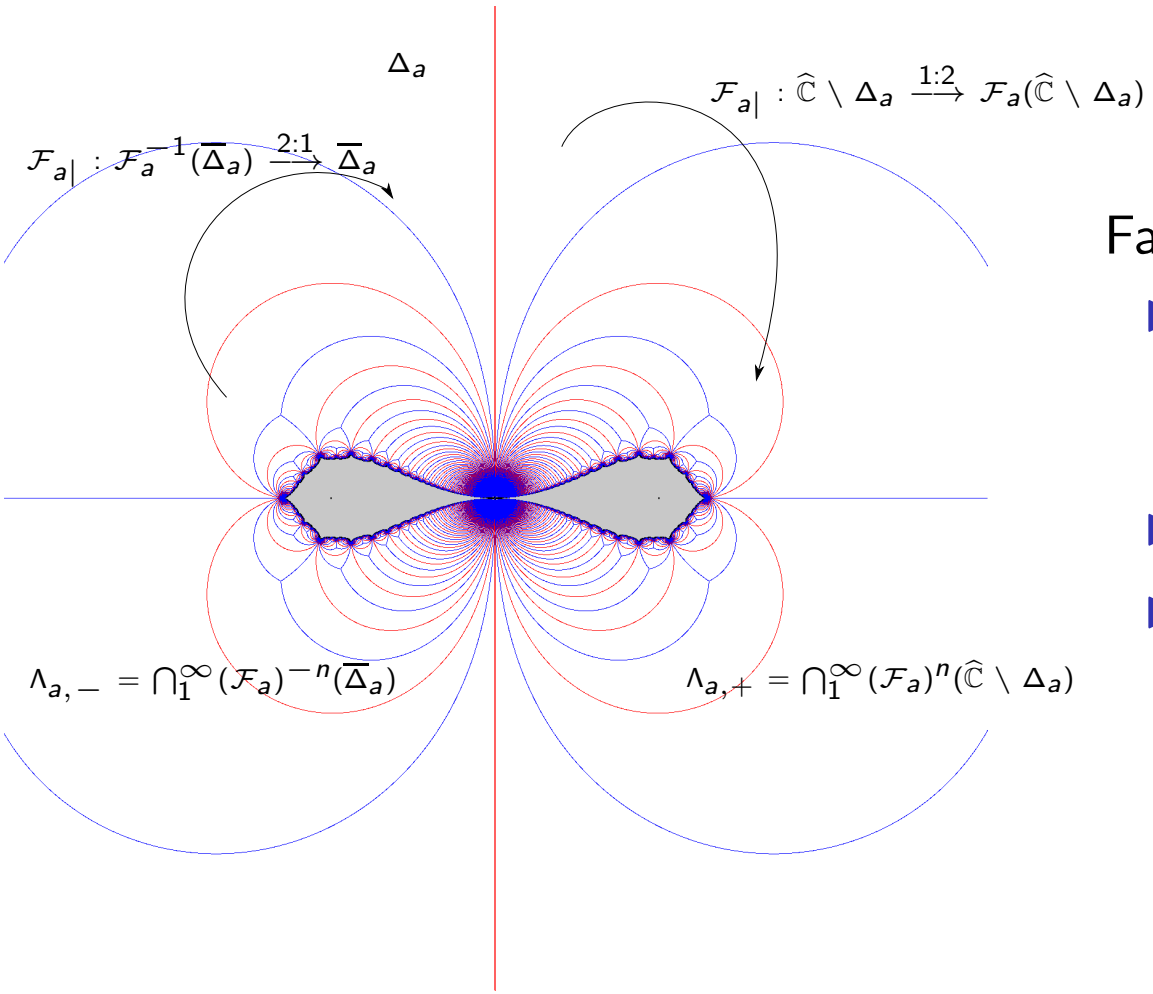
$Q(-2) = Q(1)$, so Cov_0^Q sends each blue line to the other two.







Dynamics of the family $\mathcal{F}_a$

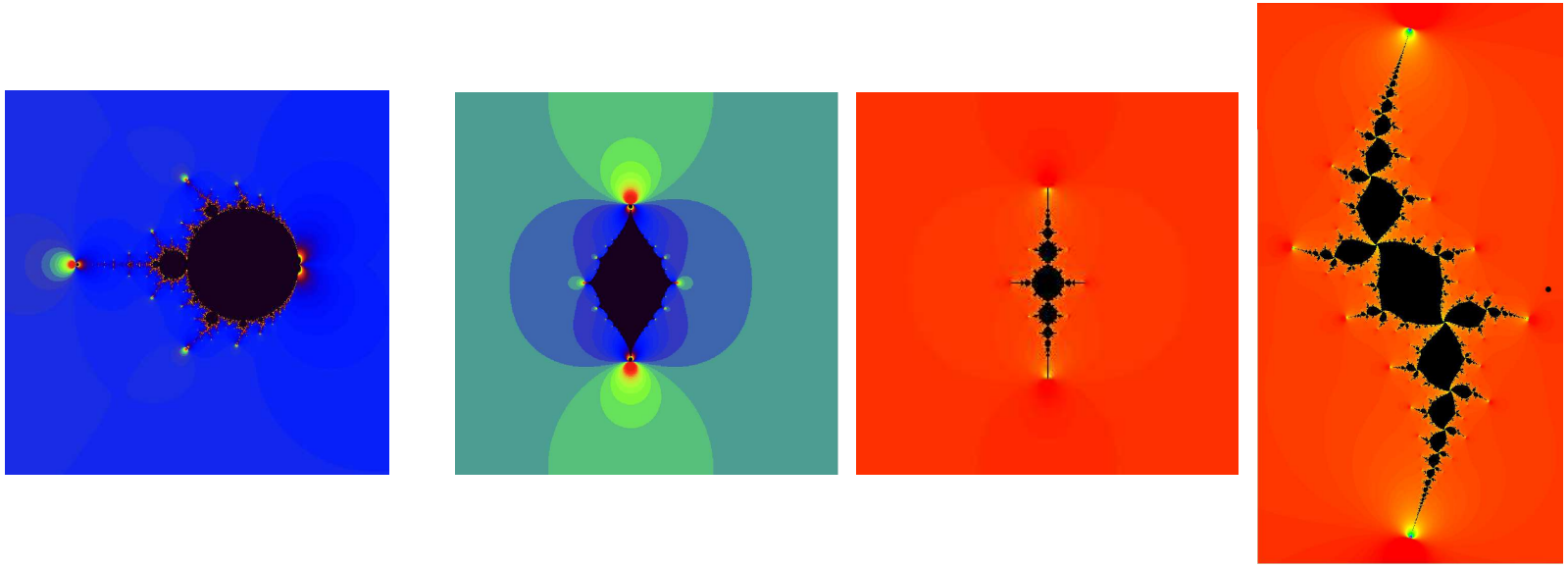


Facts: for every a

- $\mathcal{F}_a$ has a parabolic fixed point at $z = 0$.
- $\mathcal{F}_a^{-1}(0) = \{0, S_a\}$.
- $\mathcal{F}_{a|\overline{\Delta}_a}$ is a deg 2 holomorphic map at every $z \in \overline{\Delta}_a$, $z \neq S_a$

The family $\mathrm{Per}_1(1)$

- $\mathrm{Per}_1(1) = \{P_A(z) = z + 1/z + A \,|\, A \in \mathbb{C}\}$,
- ∞ parabolic fixed point, with basin Λ_A, $K_A = \widehat{\mathbb{C}} \setminus \overline{\Lambda}_A$
- $\forall A$, external class given by $h_2(z) = \frac{z^2+1/3}{1+z^2/3}$, and $h_2(z)|_{\mathbb{S}^1} \sim_{top} P_0(z) = z^2|_{\mathbb{S}^1}$
- M_1: connectedness locus ($M_1 \approx M$ by Petersen-Roesch).



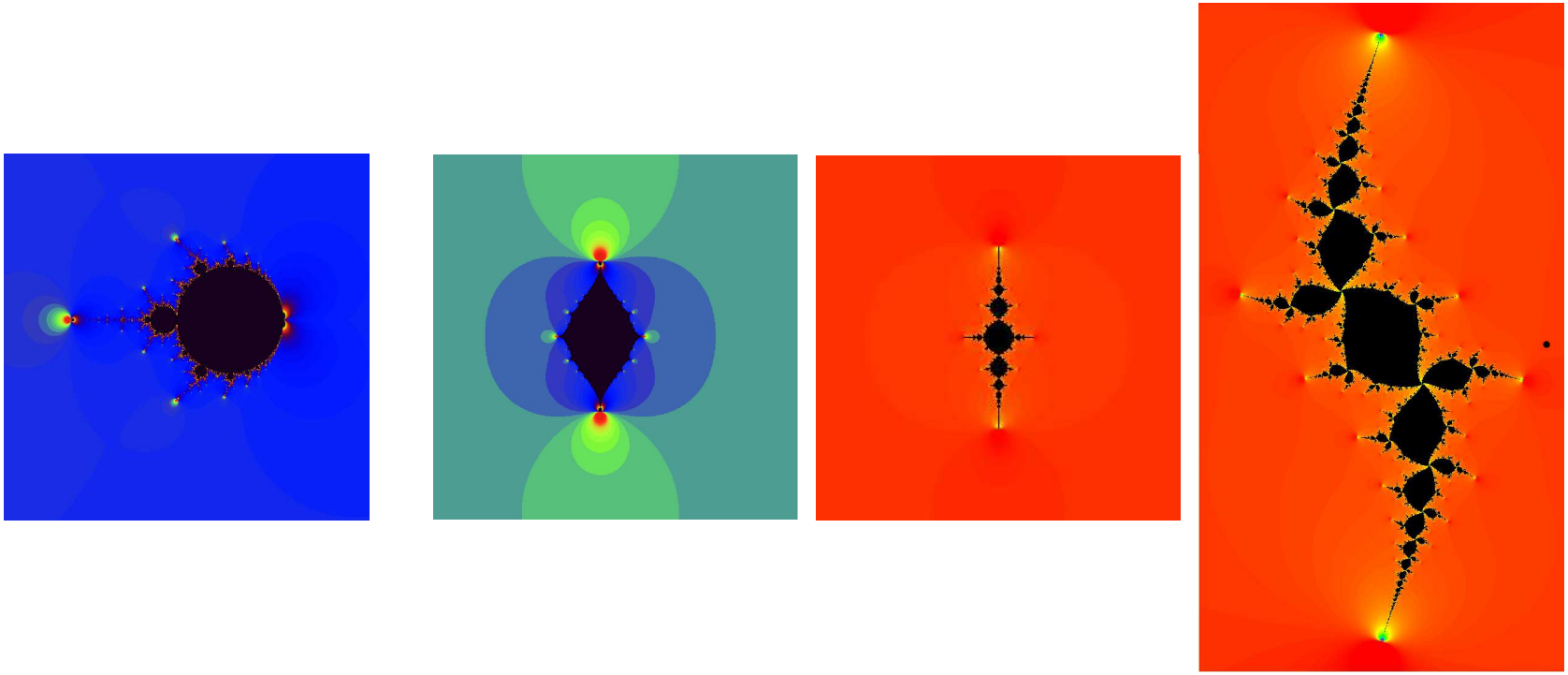
Main Theorem

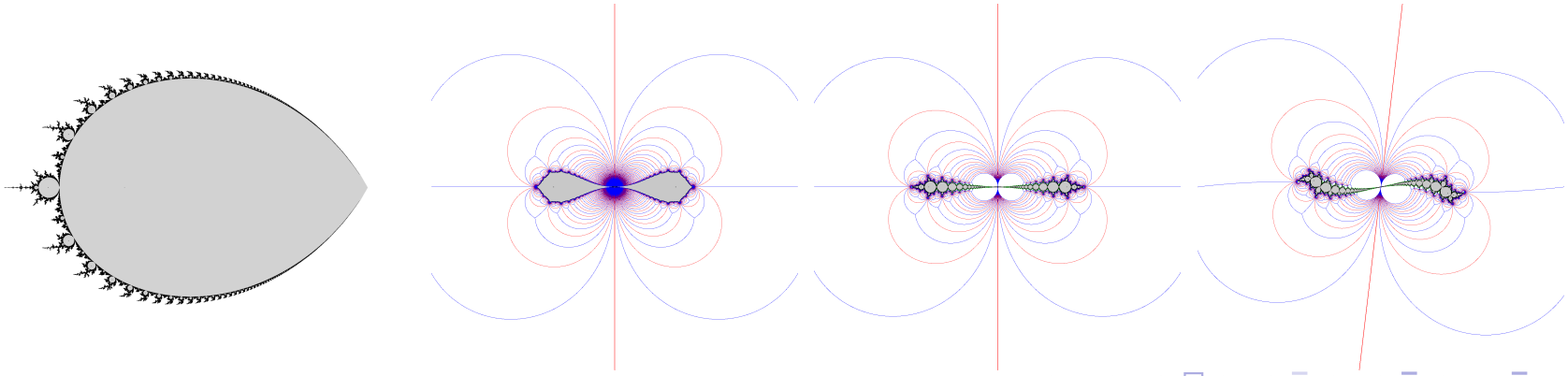
We say that $\mathcal{F}_a$ is a **mating** between the rational quadratic map $P_A : z \to z + 1/z + A$ and the modular group $\Gamma = PSL(2, \mathbb{Z})$ if

1. the 2-to-1 branch of $\mathcal{F}_a$ for which Λ_- is invariant, is hybrid equivalent to P_A on Λ_- (i.e., it is quasiconformally conjugate to P_A in a nbh of Λ_- by a map which is conformal on the interior of Λ_-),
2. when restricted to a $(2:2)$ correspondence from $\Omega(\mathcal{F}_a)$ to itself, $\mathcal{F}_a$ is conformally conjugate to the pair of Möbius transformations $\{\tau_1, \tau_2\}$ from the complex upper half plane $\mathbb{H}$ to itself.

Theorem (Bullett-L, *Invent.Math. 220, (2020)*)
For every $a \in M_\Gamma$ the correspondence $\mathcal{F}_a$ is a mating between some rational map $P_A : z \to z + 1/z + A$ and Γ.

M_1 and M_Γ





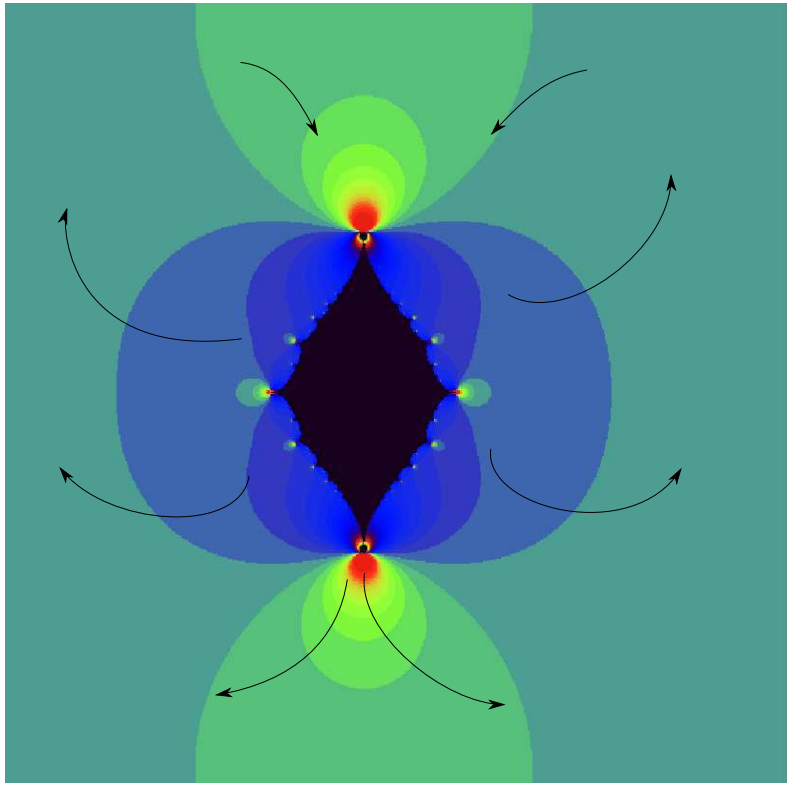
Proof of the Thm, part 1

A (deg 2) parabolic-like map is an object that locally behaves like a map $P_A, A \in \mathbb{C}$.

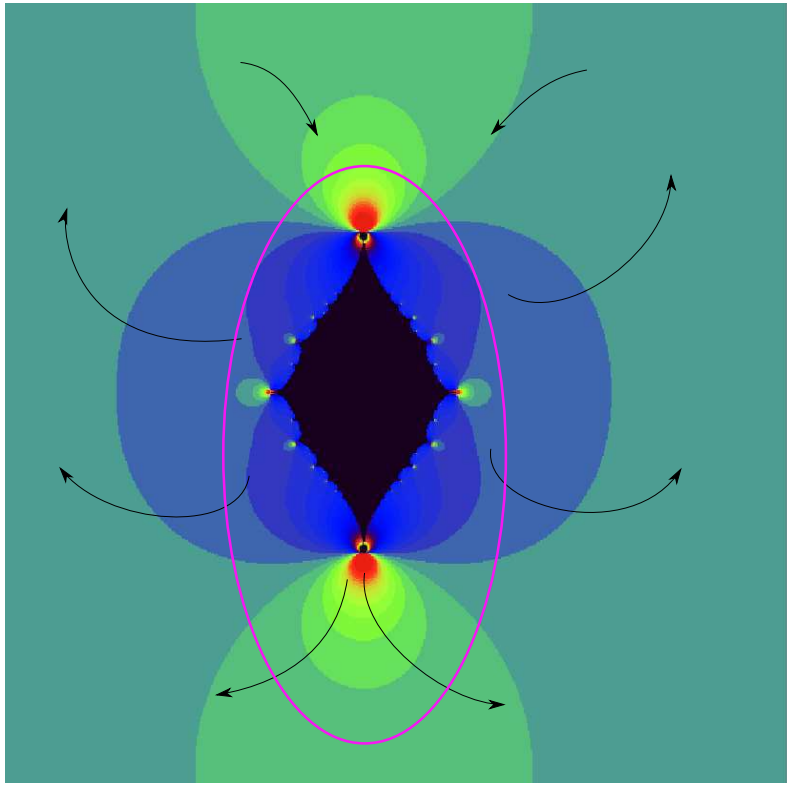
Thm A deg 2 parabolic-like map is hybrid conjugated to a member of the family $Per_1(1)$, a unique such member if the filled Julia set is connected.

So, if we prove that for all $a \in M_\Gamma$, the branch of $\mathcal{F}_a$ which fixes Λ_- restricts to a parabolic-like map, we are done!

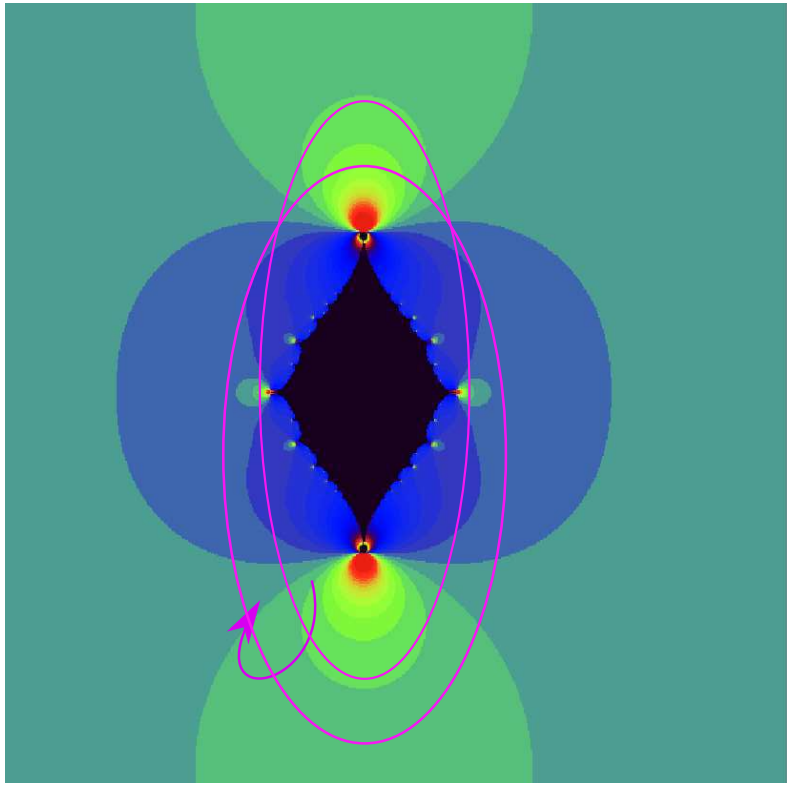
Parabolic-like maps



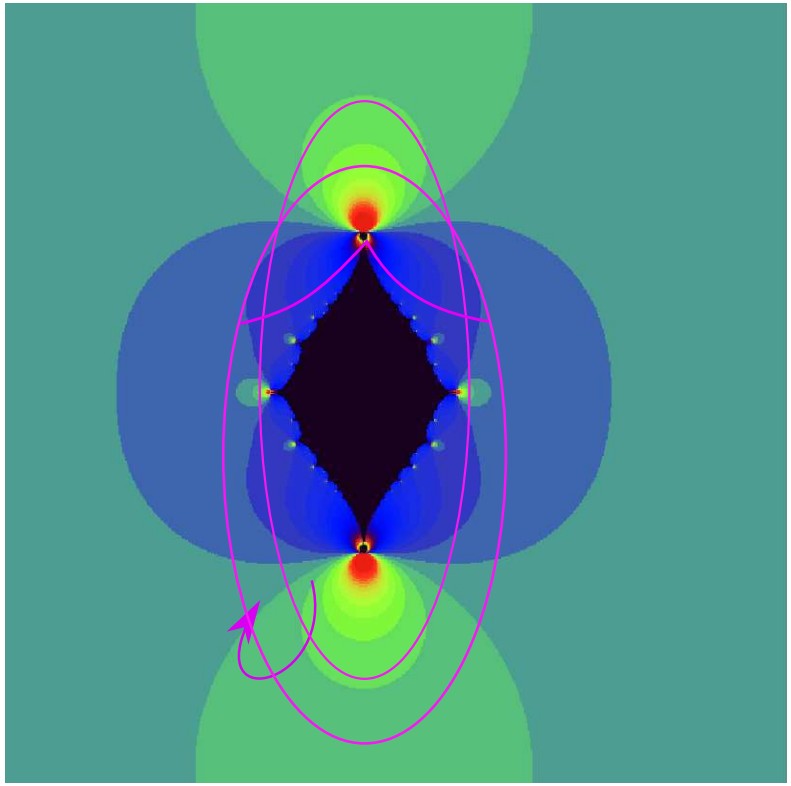
Parabolic-like maps



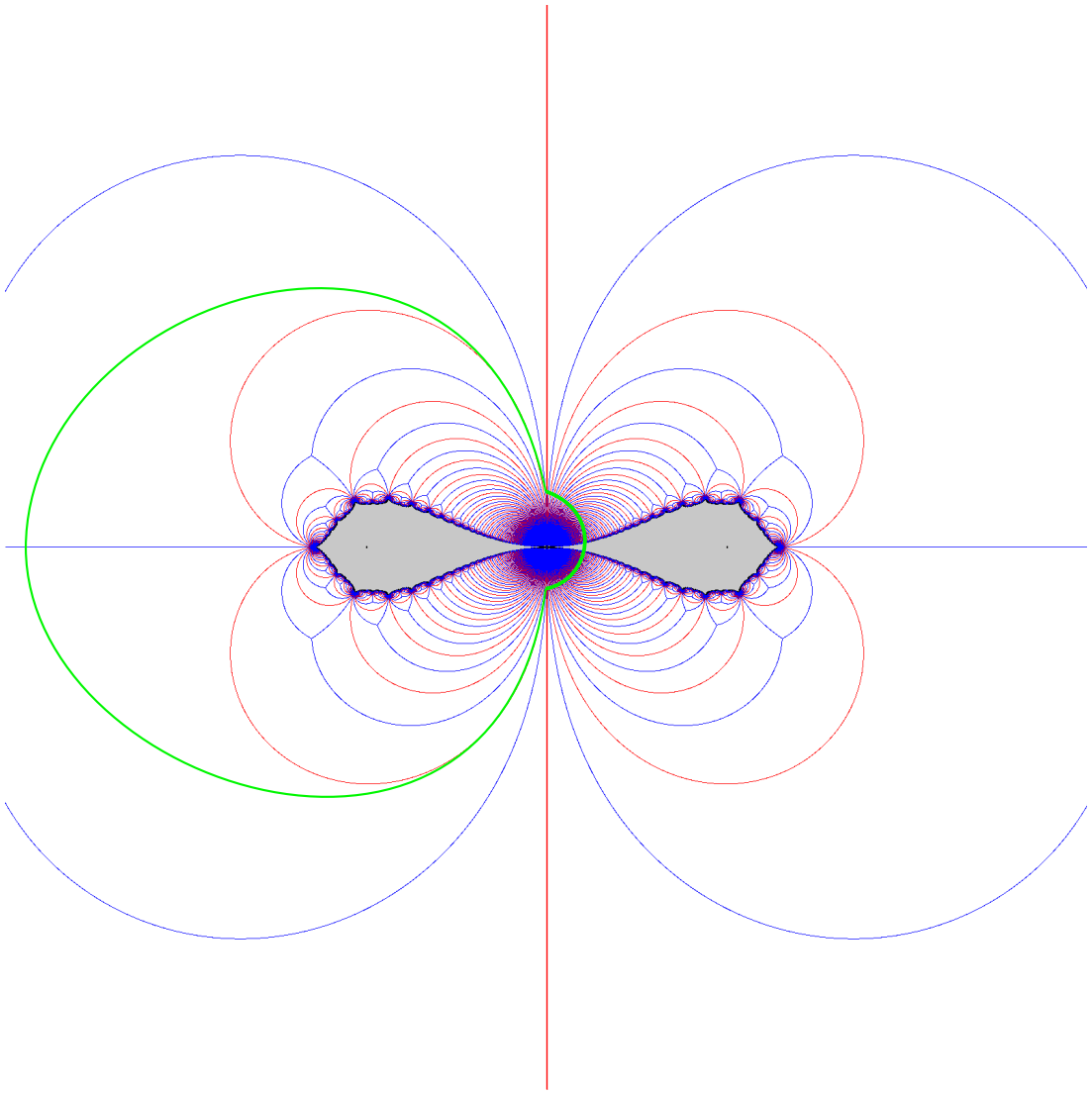
Parabolic-like maps



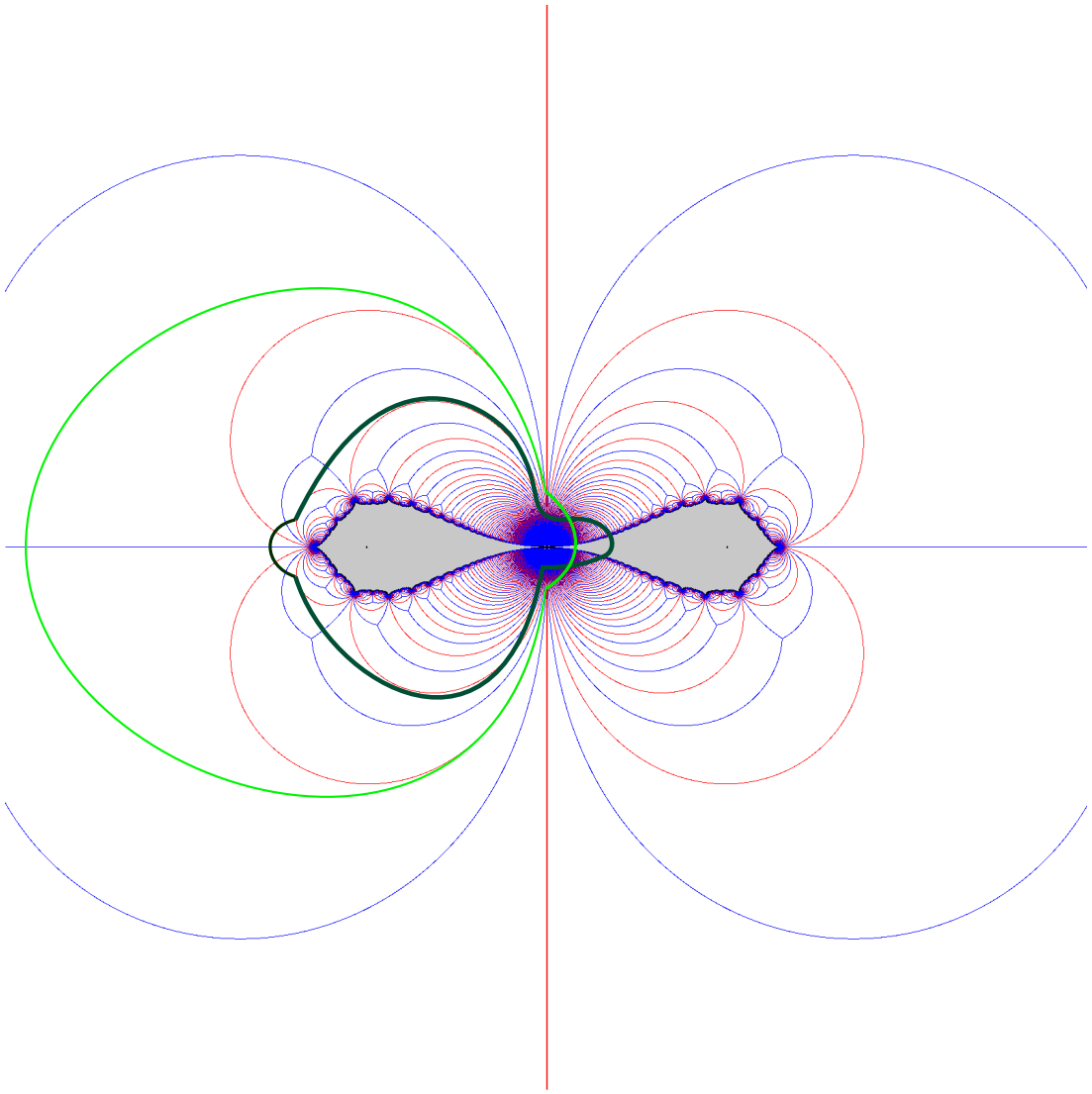
Parabolic-like maps



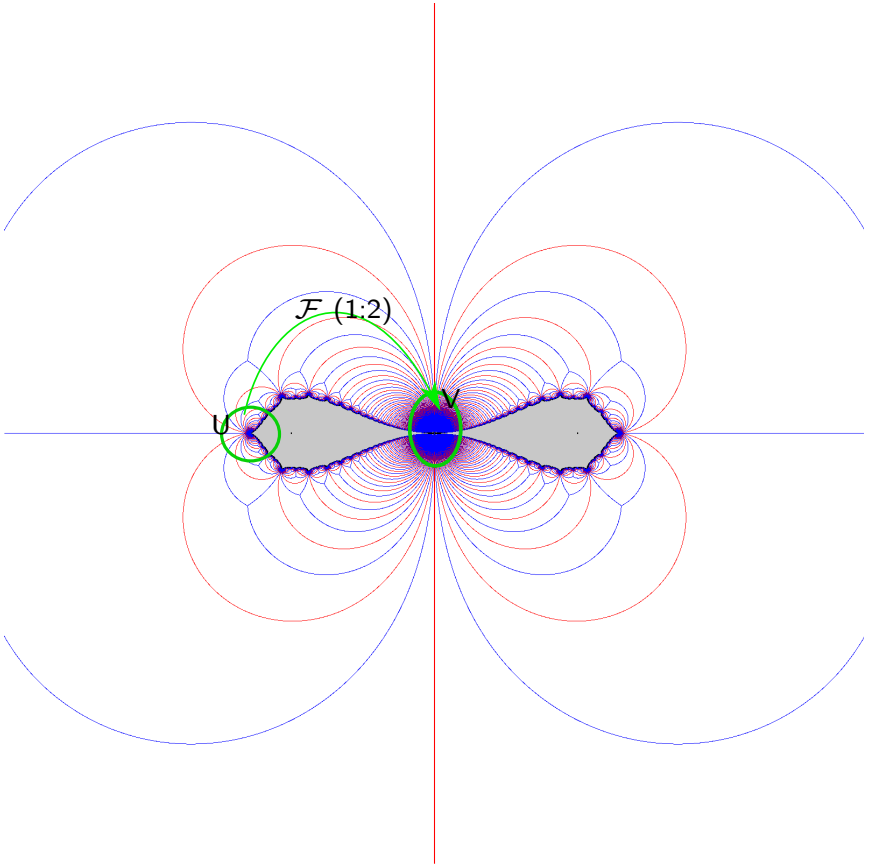
From correspondences to parabolic-like maps

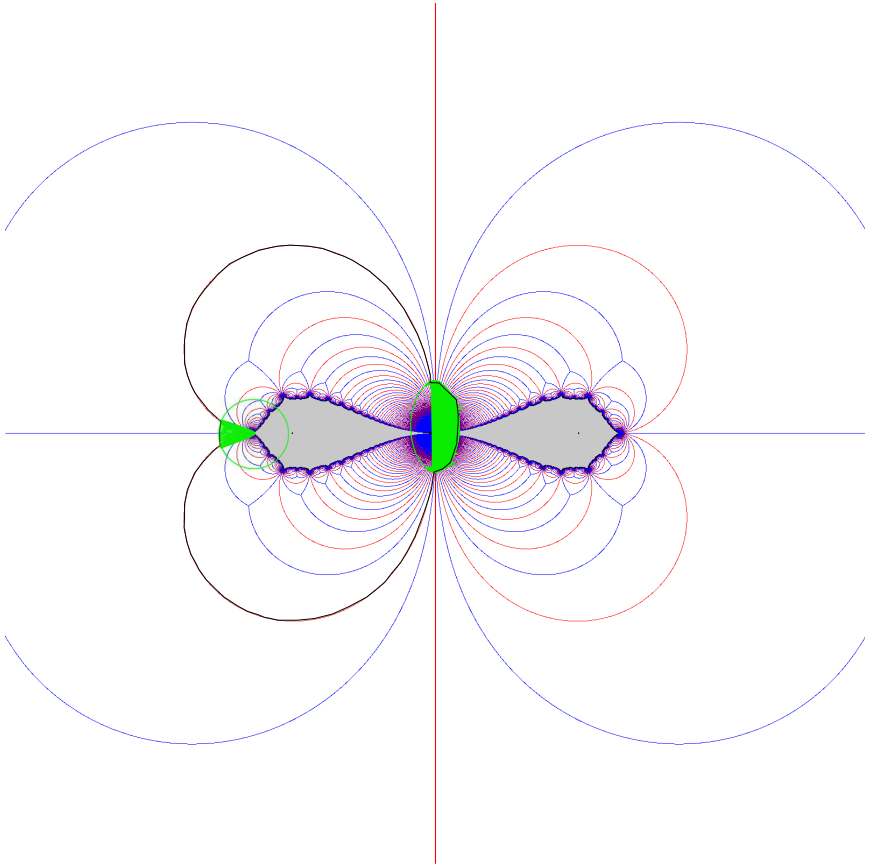


From correspondences to parabolic-like maps

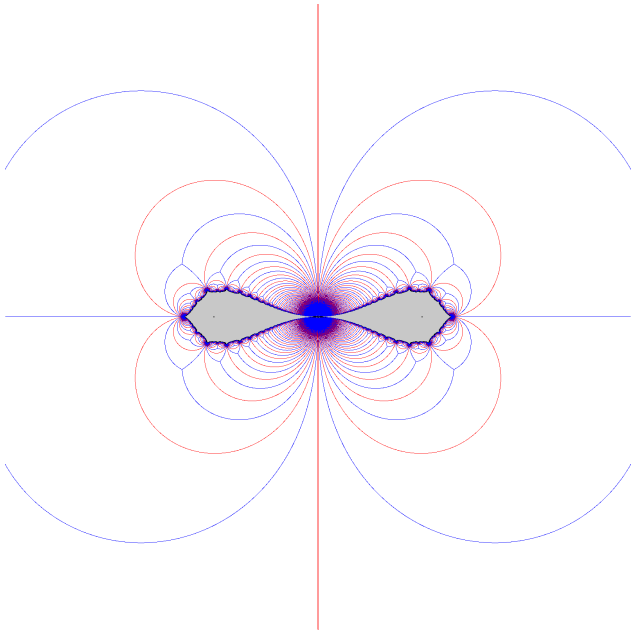


From correspondences to parabolic-like maps: qc surgery to kill one image



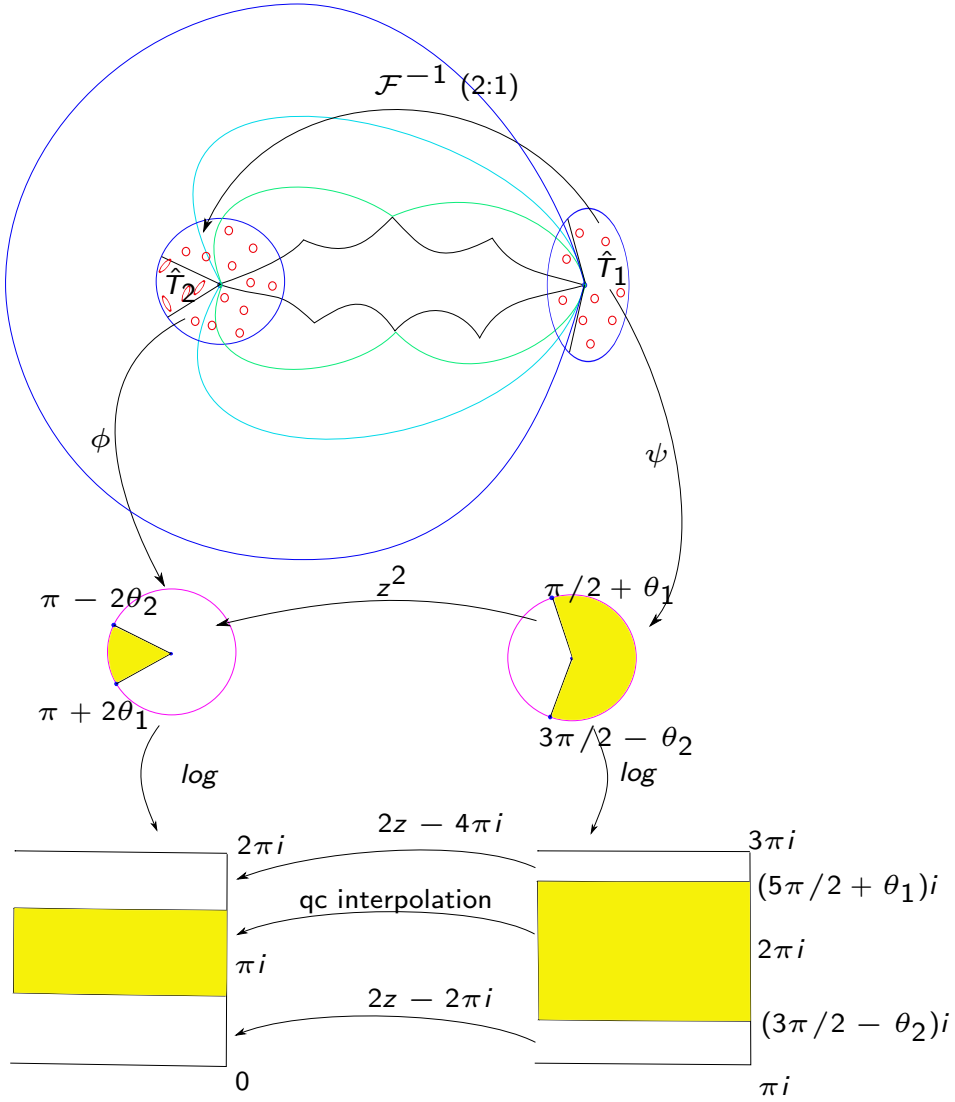


Surgery construction (valid also out of M_Γ)



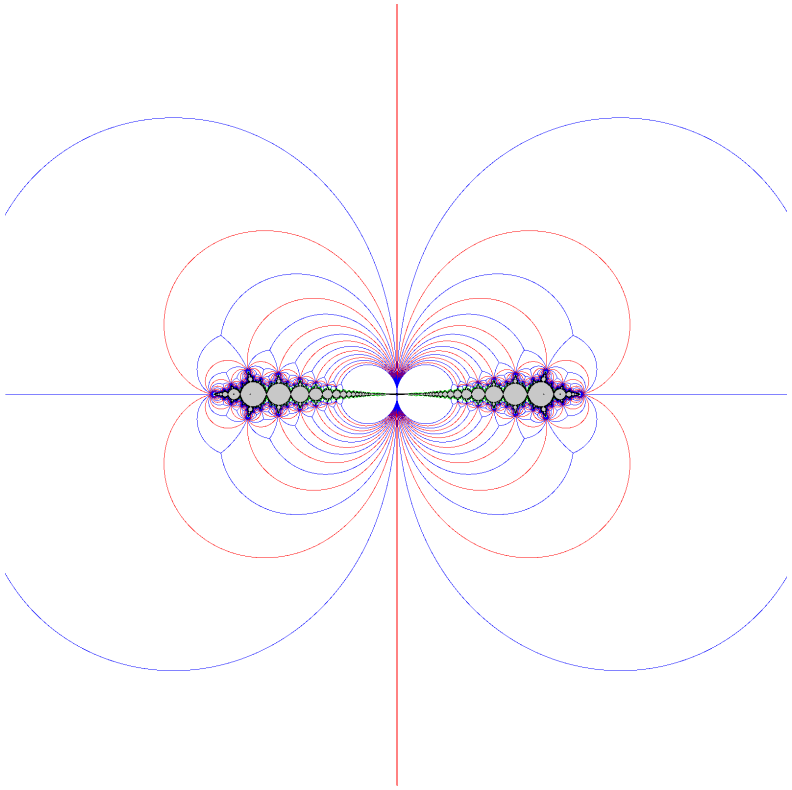
$\Lambda_- \subset \mathbb{H}_l$, but we need it to be contained in a top. disc making at the parabolic fixed point an angle $< \pi$, to have space to put our new Beltrami forms!

'Easy' to do individually

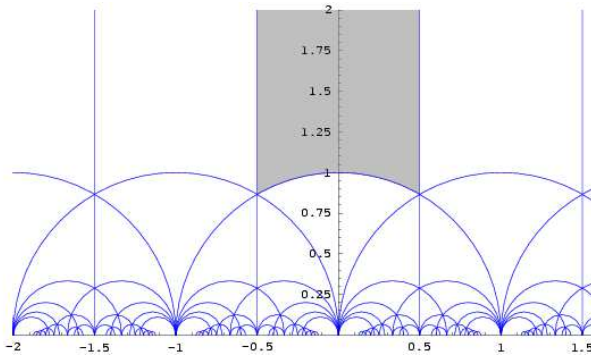


Part 2: Böttcher map (valid just in M_Γ)

- $\forall a \in M_\Gamma$ $\exists$ a Riemann map $\phi : \Omega \to \mathbb{H}$. We prove that ϕ conjugates $\mathcal{F}_{a|\Omega(\mathcal{F}_a)}$ to the generators of the modular groups on $\mathbb{H}$.
- This is: the map $\phi : \Omega \to \mathbb{H}$ plays for our family $\mathcal{F}_a$ the role the Böttcher map plays for the quadratic family!



$$\xrightarrow{\phi}$$

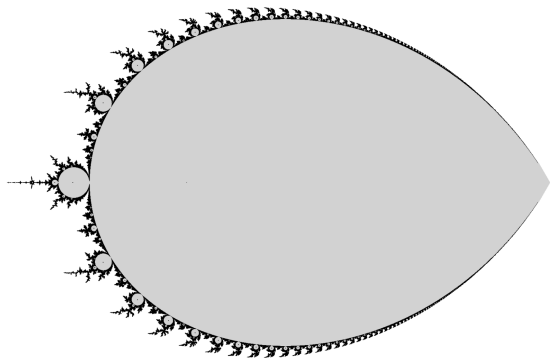


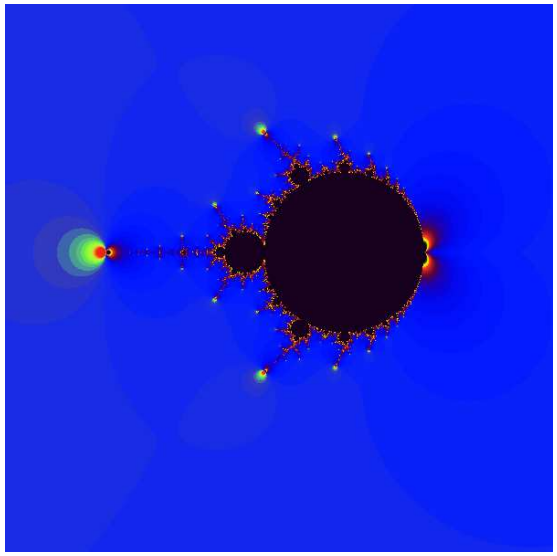
Proving $M_\Gamma \approx M_1$

The straightening construction induces a map

$$\chi : M_\Gamma \to M_1$$

The map χ is injective (punch-line: Rickmann Lemma).

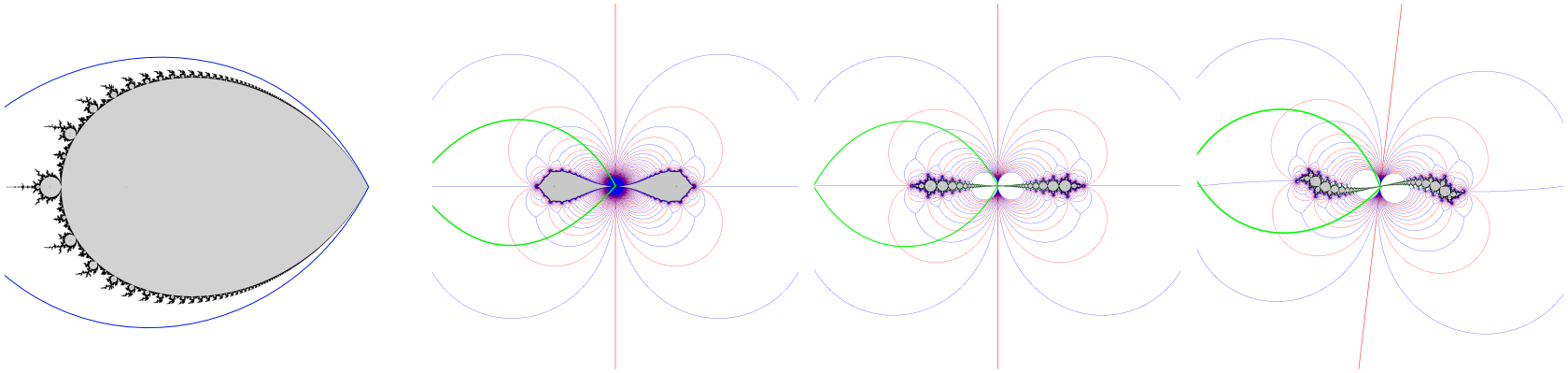




Theorem (Bullett-L) The map $\chi : M_\Gamma \setminus \{4, 7\} \to M_1 \setminus \{-3, 1\}$ is a homeomorphism, which extends to a doubly pinched neighbourhood of M_Γ.

Main tool: lunes and fundamental croissants

$M_\Gamma \subset$ lune $\mathcal{L} \Rightarrow$ for all $a \in M_\Gamma$, $\Lambda_{-,a}$ is contained in a lune V_a which moves holomorphically with the parameter.



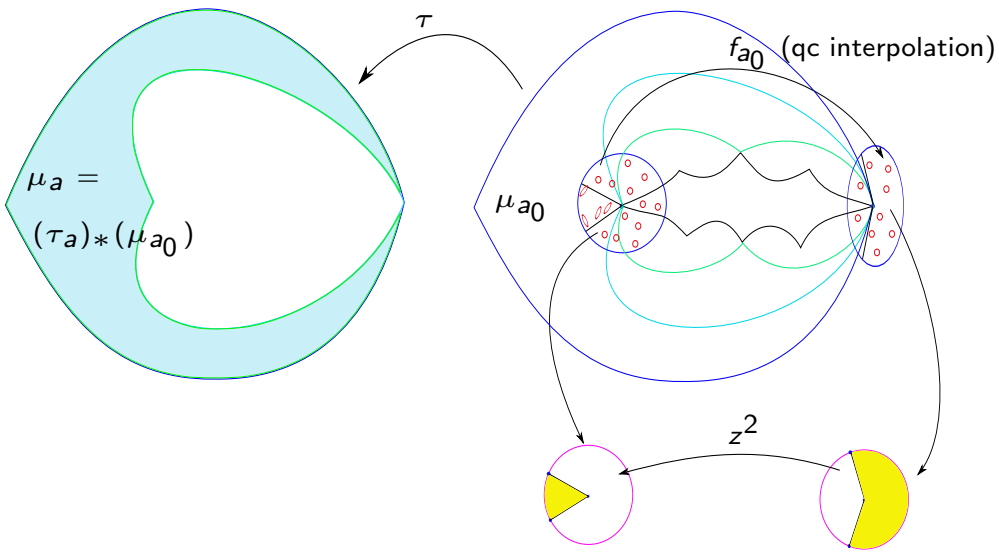
For $a \in \mathcal{L}$, set $V'_a := \mathcal{F}_a^{-1}(V_a)$. We call $\mathcal{A}_a := V_a \setminus V'_a$ the *fundamental croissant*.
The fundamental croissant moves holomorphically for $a \in \mathring{\mathcal{L}}$: fix $a_0 \in M_\Gamma \setminus \{4, 7\}$, we have a holomorphic motion

$$\tau_a : \mathcal{A}_{a_0} \to \mathcal{A}_a$$

We can extend $\chi : \mathring{\mathcal{L}} \setminus M_\Gamma \to \mathbb{C}$ by using τ: the extension is qc.

Holomorphic motion of Beltrami forms

Do surgery at $a_0 \in M_\Gamma$, obtaining μ_{a_0} on $\mathcal{A}_{a_0}$, move it by $\tau : \mathcal{A}_{a_0} \to \mathcal{A}_a$ $\mu_a = (\tau_a)_*(\mu_{a_0})$, and $\overline{\mu}_a = (\mathcal{F}_a)^{-n}(\mu_a)$ on $\mathcal{F}_a^{-n}(\mathcal{A}_a)$ for all $n \geq 1 \Rightarrow \overline{\mu}_a$ is holomorphic in a wherever $\Lambda_{a,-}$ moves holomorphically with a.



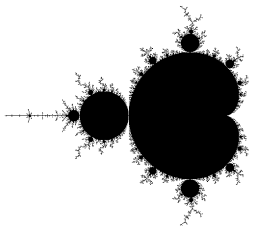
Technical: we can construct γ_a holomorphic with a for a in a doubly pinched neighbourhood $U(M_\Gamma)$ of M_Γ (pinched at 4 and 7)

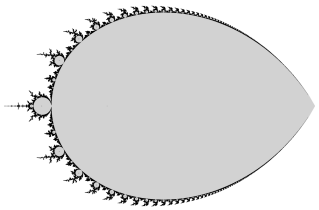
Last steps

- $U(M_\Gamma) \setminus \partial M_\Gamma$ is the set of parameters where $\Lambda_{a,-}$ moves holomorphically (by MSS decomposition).
- χ holomorphic on $\mathring{M}_\Gamma$ by holomorphic motion arguments
- $+$ qc. out of M_Γ $\Rightarrow \chi$ continuous on $U(M_\Gamma) \setminus \partial M_\Gamma$
- sequences of qc maps have convergent subs. $+$ rigidity on ∂M_1 $\Rightarrow \chi$ continuous on ∂M_Γ.
- χ is a branched covering, and as it is injective on M_Γ, it is a homeo on $U(M_\Gamma)$.

Dictionary between Q_c and $\mathcal{F}_a$

Quadratic polynomials Q_c	*Quadratic correspondences* $\mathcal{F}_a$
$\varphi_c : \hat{\mathbb{C}} \setminus K(Q_c) \cong \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$	$\varphi_a : \hat{\mathbb{C}} \setminus \Lambda(\mathcal{F}_a) \cong \mathbb{H}$
external rays	external geodesics
'periodic rays land'	'periodic geodesics land'
every repelling fixed point is the landing point of a periodic ray	every repelling fixed point is the landing point of a periodic geodesic
Yoccoz inequality $\sim 1/q$	Yoccoz inequality $\sim (\log q)/q^2$
$M \subset \overline{D(0,2)}$	$M_\Gamma \subset \mathcal{L}_\theta$





Thanks for your attention!