THE CUBIC CONNECTEDNESS LOCUS: LEMON LIMBS

Carsten Lunde Petersen, IMFUFA at INM Roskilde University
joint work with Saeed Zakeri CUNY

On the Geometric Complexity of Julia Sets - II
On-line Conference August 2020 hosted by
The Stefan Banach International Mathematical Center
GUIDING QUESTION.

- Limb decomposition has proven to be a very effective way to analyse the Mandelbrot set.
- Is there a fruitfull similar divide and conquer approach to understanding the cubic connectedness locus $C$?
- Does the cubic connectedness locus also have a tangible limb structure?
We will be working with cubic polynomials in the normal form

\[ P_{a,b} : z \mapsto z^3 + 3az^2 + b \quad a, b \in \mathbb{C} \quad (1) \]

with marked critical points at 0, \(-2a\). More precisely, we have

<table>
<thead>
<tr>
<th>critical points</th>
<th>0</th>
<th>(-2a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>critical values</td>
<td>(b)</td>
<td>(4a^3 + b)</td>
</tr>
<tr>
<td>co-critical points</td>
<td>(-3a)</td>
<td>(a)</td>
</tr>
</tbody>
</table>
Basic notation I.

- We will think of 0 and $-2a$ as the “first” and “second” critical points, respectively.
- The space $\mathcal{P}(3)$ of all such critically marked cubics is isomorphic to $\mathbb{C}^2$ with coordinates $(a, b)$.
- Two distinct cubics $P_{a,b}$ and $P_{a',b'}$ are conjugate by an affine map preserving the marking if and only if $a' = -a$, $b' = -b$. The unique such conjugacy is $\tau(z) = -z$.
- Thus, the space of all affine conjugacy classes (the so-called “moduli space”) of marked cubics is also isomorphic to $\mathbb{C}^2$ with coordinates $(a^2, b^2)$.
- Note however that conjugacy by $\tau$ changes the labelling of external rays, i.e it shifts all arguments by $\frac{1}{2}$ mod 1.
Basic notation II.

- $Q_c : z \mapsto z^2 + c$, with Julia set and filled Julia set $J_c$ and $K_c$, Böttcher coordinate $\phi_c$, external rays $R_c(\theta)$

- $M$: the Mandelbrot set, with external rays $R_M(\theta)$

- $P_{a,b}(z) = z^3 + 3az^2 + b$, with marked critical points at 0 and $-2a$, filled Julia set and Julia set $K_{a,b}$ and $J_{a,b}$, Böttcher coordinate $\phi_{a,b}$, external rays $R_P(\theta)$

- $P(3) \cong \mathbb{C}^2, C(3), \mathcal{E}(3)$: The cubic parameter space, connected locus and escape locus

- $H$: The principal hyperbolic component in $C(3)$ containing $P_{0,0} : z \mapsto z^3$. 
The Lemon family – The starting point –.

- We call the one complex dimensional family
  \[ P_a(z) = P_{a,0} = z^3 + 3az^2, \quad a \in \mathbb{C} \]
  the Lemon Family.

- The point 0 is a persistent super attracting fixed point in the Lemon family. We denote by \( \Lambda_a \) the immediate basin of 0 for \( P_a \).

- The central hyperbolic component \( \mathcal{H}_0 \) of the Lemon family is
  \[ \mathcal{H}_0 := \{ a \mid -2a \in \Lambda_a \} \]

- By the Faught-Roesch theorem
  1. the immediate basin \( \Lambda_a \) is a Jordan domain, when \( a \notin \mathcal{H}_0 \).
  2. \( \mathcal{H}_0 \) is a Jordan domain and \( P_a \) has a quadratic-like restriction hybridly equivalent to \( Q_0 \) and with filled-in Julia set \( \overline{\Lambda}_a \) if and only if \( a \notin \overline{\mathcal{H}}_0 \).
The Lemon $\mathcal{H}_0$ and some of its limbs.
We say that $P = P_{a,b} \in \mathcal{P}(3)$ is **centrally renormalizable** if $P$ has a quadratic-like restriction $P : U \to U'$ with $0 \in U$ and with connected filled Julia set. Thus $P^n(0) \in U$ for all $n$ while $-2a \notin U$. 
The central renormalization curves

Definition
Let $c \in \mathcal{M}$ and define $\mathcal{X}_c \subset \mathcal{P}(3)$ as the set of all centrally renormalizable $P_{a,b}$, whose quadratic-like restriction around 0 is hybrid equivalent to $Q_c$.

- By definition, the sets $\mathcal{X}_c$ for distinct $c$ are disjoint.
- $\mathcal{X}_0 = \mathbb{C} \setminus \overline{\mathcal{H}_0}$

Lemma
$\mathcal{X}_c$ is a one-dimensional complex analytic set for every $c \in \mathcal{M}$, that is, every point in $\mathcal{X}_c$ has an open neighborhood $U \subset \mathcal{P}(3)$ such that $\mathcal{X}_c \cap U$ is biholomorphic to the disk $\mathbb{D}$. 
The escape region and $\mathcal{X}_c$.

**Definition**

We denote by $\mathcal{E}_c$ the set of parameters in $\mathcal{X}_c$ for which the second critical point $-2a$ escapes to $\infty$

$$\mathcal{E}_c = \mathcal{X}_c \cap \mathcal{E}(3).$$

**Theorem (Branner-Hubbard)**

The set $\mathcal{E}_c$ is isomorphic to the punctured disk $\mathbb{C} \setminus \overline{D}$.

$\mathcal{E}_c$ will henceforth be denoted the escape region in $\mathcal{X}_c$.

All cubics in $\mathcal{E}_c$ are quasiconformally conjugate.

**Question**

Is $\mathcal{X}_c$ connected? Is it embedded in $\mathcal{P}(3)$? Is $\pi_1(\mathcal{X}_c)$ infinite cyclic generated by the loop around the puncture in $\mathcal{E}_c$?
Examples of $\mathcal{E}_c$. 

Name: cubicsuperper2.so  
XLeft: -1.29844  
XRight: 1.54219  
Iteration: 100  
ColorMap: default.map  
No parameters

Name: cubicsuperper3torusray.so  
XLeft: 0.0726171  
XRight: 1.51977  
Iteration: 100  
ColorMap: default.map  
No parameters

Name: cubicsuperper3torusray.so  
XLeft: 1.29762  
XRight: 2.09246  
Iteration: 100  
ColorMap: default.map  
No parameters

Name: cubicsuperper3torusray.so  
XLeft: 1.29762  
XRight: 2.09246  
Iteration: 100  
ColorMap: default.map  
No parameters
Combinatorics of periodic orbits.

- Let \( f : \mathbb{T} \to \mathbb{T} := \mathbb{R}/\mathbb{Z} \) be a degree \( k \geq 2 \) covering map.

- Example: the multiplication-by-\( k \) map \( m_k : t \mapsto kt \pmod{\mathbb{Z}} \).

- Let \( \mathcal{O} = \{t_1, \ldots, t_q\} \) with \( 0 \leq t_1 < \cdots < t_q < 1 \) be a \( q \)-periodic orbit for \( f \).

- Example \( f = m_2 \) and \( \mathcal{O} = \{1/5, 2/5, 3/5, 4/5\} \) with \( k = 2 \) and \( q = 4 \).

- By the **combinatorics** of \( \mathcal{O} \) under \( f \) we mean the cyclic permutation \( \sigma \) of \( \{1, \ldots, q\} \) defined by

\[
    f(t_j) = t_{\sigma(j)} \quad \text{for all } 1 \leq j \leq q.
\]

- in the example above \( \sigma = (1, 2, 4, 3) \).
Illustrations.
Theorem

A given combinatorics $\sigma$ is realised at most once by $m_2$.

Theorem

Every $q$-periodic combinatorics $\sigma$ realised by $m_2$ is realised precisely $q + 1$ times under $m_3$ by orbits $O_0, \ldots, O_q$. For $q > 1$ these orbits are uniquely determined by the deployment condition

$$#(O_j \cap [0, 1/2]) = j \quad 0 \leq j \leq q.$$
Orbit interlacing and simulating orbits

**Theorem**

Let \( \{ t_1, \ldots, t_q \} \) be an \( m_2 \) orbit of period \( q \geq 2 \) and combinatorics \( \sigma \). For each \( 1 \leq k \leq q \), the neighboring orbits \( O_{k-1}, O_k \) interlace. More precisely, if \( O_k = \{ x_1, \ldots, x_q \} \) and \( O_{k-1} = \{ y_1, \ldots, y_q \} \), then

\[
0 < x_1 < y_1 < \cdots < x_k < \frac{1}{2} < y_k < \cdots < x_q < y_q < 1. \quad (3)
\]

**Definition**

We call the pair

\[
O^-(t) := O_k \quad \text{and} \quad O^+(t) := O_{k-1}
\]

the **simulating orbits** for the periodic point \( t = t_k \).

We extend the above definitions to the fixed point \( t = 0 \) by setting

\[
O^-(0) := \{ 0 \} \quad \text{and} \quad O^+(0) := \{ 1/2 \}.
\]
Simulating orbits.

- We call the pair points

\[ x_k \in \mathcal{O}^-(t) \quad y_k \in \mathcal{O}^+(t) \]

the **critical angles** associated with \( t \) and

- \( m_3(x_k) = x_{\sigma(k)} \quad m_3(y_k) = y_{\sigma(k)} \)

the **critical value angles** associated with \( t \).

- An easy computation shows that

\[ y_k - x_k = \frac{3q - 1}{3q - 1} \quad \text{and} \quad y_{\sigma(k)} - x_{\sigma(k)} = \frac{1}{3q - 1}. \]
rabbit

\[ \theta^+ = \frac{15}{26} \]

\[ \theta^- = \frac{6}{26}, \frac{5}{26} \]

\[ \mathcal{O}_1 \cup \mathcal{O}_2 \]

\[ \theta^+ = \frac{14}{26} \]

\[ \theta^- = \frac{5}{26} \]

\[ \mathcal{O}_0 \cup \mathcal{O}_1 \]

\[ t_1 = \frac{1}{7} \]

\[ t_2 = \frac{2}{7} \]

\[ t_3 = \frac{4}{7} \]

\[ \sigma = (1, 2, 3) \]
Cocapelli

\[
\begin{align*}
\theta^- &= \frac{24}{80}, \quad \theta^+ = \frac{51}{80} \\
\theta^- &= \frac{17}{80}, \quad \theta^+ = \frac{56}{80} \\
\theta^- &= \frac{8}{80}, \quad \theta^+ = \frac{73}{80} \\
\theta^- &= \frac{1}{80}, \quad \theta^+ = \frac{21}{80} \\
\theta^- &= \frac{2}{80}, \quad \theta^+ = \frac{24}{80} \\
\theta^- &= \frac{21}{80}, \quad \theta^+ = \frac{56}{80} \\
\theta^- &= \frac{7}{80}, \quad \theta^+ = \frac{63}{80} \\
\theta^- &= \frac{4}{80}, \quad \theta^+ = \frac{68}{80} \\
\theta^- &= \frac{36}{80}, \quad \theta^+ = \frac{72}{80} \\
\theta^- &= \frac{28}{80}, \quad \theta^+ = \frac{21}{80} \\
\theta^- &= \frac{12}{80}, \quad \theta^+ = \frac{36}{80} \\
\theta^- &= \frac{7}{80}, \quad \theta^+ = \frac{28}{80} \\
\theta^- &= \frac{4}{80}, \quad \theta^+ = \frac{7}{80} \\
\theta^- &= \frac{1}{80}, \quad \theta^+ = \frac{1}{80}
\end{align*}
\]

\[\sigma = (1, 2, 4, 3)\]
Lemon Limbs and Wakes I.

Definition

Let \( t = t_k \) be the \( k \)-th point in a \( q \)-cycle for \( m_2 \) as above and let \( O_{k-1} = \{x_1, \ldots, x_q\} \) and \( O_k = \{y_1, \ldots, y_q\} \) be the associated simulating cycles for \( m_3 \). We define the \( t \)-wake \( \mathcal{W}(t) \subset \mathcal{P}(3) \) as

\[
\mathcal{W}(t) := \{ P \in \mathcal{P}(3) : R_P(x_j) \text{ and } R_P(y_j) \text{ co-land for } j = 1, \ldots q \}
\]

and the corresponding "Lemon" \( t \)-limb

\[
\mathcal{L}(t) = C(3) \cap \mathcal{W}(t)
\]

Note that the points \( z_j(P) = \overline{R_P(x_j)} \cap \overline{R_P(y_j)} \) form a periodic orbit whose period is a divisor of \( q \) and that either this orbit is repelling or \((P^q)'(z_j(P)) = 1\). Also note that wakes are neither open nor closed.
The involution $\theta \mapsto \theta^* := \theta + \frac{1}{2}$ is an automorphism of $\mathfrak{m}_3$ exchanging the fixed points 0 and $\frac{1}{2}$ for $\mathfrak{m}_3$. We define

$$\mathcal{W}^*(t) := \{ P \in \mathcal{P}(3) : R_P(x^*_j) \text{ and } R_P(y^*_j) \text{ co-land for } j = 1, \ldots, q\}$$

and

$$\mathcal{L}^*(t) = \mathcal{C}(3) \cap \mathcal{W}^*(t_k)$$

Then $P$ belongs to $\mathcal{W}(t)$ or $\mathcal{L}(t)$ if and only if $\tau \circ P \circ \tau$ belongs to $\mathcal{W}^*(t)$ or $\mathcal{L}^*(t)$ respectively (recall $\tau(z) = -z$).

**Definition**

For $c \in \mathcal{M}$ denote by $\mathcal{L}_c(t)$ and $\mathcal{L}^*_c(t)$ the intersections

$$\mathcal{L}_c(t) := \overline{\mathcal{X}_c} \cap \mathcal{L}(t) \quad \mathcal{L}^*_c(t) := \overline{\mathcal{X}_c} \cap \mathcal{L}^*(t).$$
Wakes and Limbs in $X_0$. 

\[ \mathcal{W} \]

\[ \mathcal{W}^{(1/3)} \]

\[ \mathcal{W}^{(1/2)} \]

\[ \mathcal{W}(0) \]

\[ \mathcal{W}(1/3) \]

\[ \mathcal{W}(2/3) \]

\[ \mathcal{W}^{(2/3)} \]
More $\mathcal{E}_c$’s and corresponding conjectural $\mathcal{X}_c$’s.
What can we say about the Lemon $t$-limb for $t$ a periodic point for $m_2$?

What can we say about the intersection $\mathcal{L}_c(t)$ of the Lemon $t$-limb with $\mathcal{X}_c$ for $t$ a periodic point for $m_2$?

Is

$$\mathcal{X}_c = \mathcal{E}_c \cup_{t, t^* \text{ periodic for } m_2} \mathcal{L}_c(t) \cup \mathcal{L}_c(t^*)$$

We know from the Faught-Roesch Theorem that for $c = 0$ the answer to this question is yes.
Proposition

Let $0 \leq t \leq s < 1$ be two periodic points for $m_2$. Then

1. $\mathcal{W}(t) \cap \mathcal{W}(s) \neq \emptyset \implies t = s$.

2. For $c \in M$

   $$\mathcal{W}_c(t) \cap \mathcal{W}_c^*(s) \neq \emptyset \implies$$
   - $t = s = 0$ or $t = t_k, s = t_j$ belongs to the same rotation cycle
   - $0 < t_1 < \ldots < t_q < 1$, $\sigma = (1, 2, \ldots, q)^p$ where $(p, q) = 1$ and $k + j = q + 1$.

For $c \notin M$ the intersection possibilities of $\mathcal{W}(t)$ and $\mathcal{W}^*(s)$ is much richer.
Basic Limb dynamics

- Let \( t = t_k \) belong to a \( q \)-cycle \( 0 < t_1 < \ldots < t_q < 1 \) for \( m_2 \) with combinatorics \( \sigma \), let \( c \in M \) and let \( P = P_{a,b} \in \mathcal{L}_c(t) \).

- It is not difficult to see that the second critical point \( C = -2a \) belongs to the dynamical wake \( \mathcal{W}_P(t_k) \) with boundary rays \( \mathcal{R}_P(x_k) \) and \( \mathcal{R}_P(y_k) \).

- and more generally for each \( j = 1 \ldots q \) the iterate \( P^j(C) \) belongs to the wake \( \mathcal{W}_P(t_{\sigma^j(k)}) \) with boundary rays \( \mathcal{R}_P(x_{\sigma^j(k)}) \) and \( \mathcal{R}_P(y_{\sigma^j(k)}) \).
We say that $P \in \mathcal{L}_c(t)$ is **peripherally renormaliseable** iff $P^q$ has a quadratic-like restriction

$$P^q : V \rightarrow V'$$

with connected filled-in Julia set containing the second or peripheral critical point $C = -2a$. We call such restriction a **peripheral renormalisation** of $P$. 
Mandelbrot copies corresponding to peripheral renormalization in $\mathcal{M}_0$. 
Main Theorem

Theorem

Let $t$ be periodic for $m_2$ with orbit $0 < t_1 < \ldots t = t_k, \ldots < t_q < 1$. Let $c \in M \setminus \{\mathcal{R}_M(t_1), \ldots, \mathcal{R}_M(t_q)\}$ and let $d \in M \setminus \{1/4\}$. Then there exists a unique cubic polynomial $P = P_{a,b} \in \mathcal{L}_c(t) = \mathcal{L}_c \cap \mathcal{L}(t)$ with peripheral renormalisation $P^q : V \to V'$ hybridly equivalent to $Q_d$. 

Carsten Lunde Petersen, INM RUC
Start from the quadratic polynomial $Q_c$ and the unique cubic polynomial $\hat{P} = P_a \in \text{Per}_1(0) \cap \mathcal{L}(t_k)$, which is peripherally renormalizeable with peripheral renormalisation hybridly equivalent to $Q_d$. Construct a new holomorphic map $f : \mathbb{C} \setminus \Lambda' \to \mathbb{C}$ by changing the external class $Q_0$ of $Q_c$ to the external class of the quadratic-like map (in the sense of McMullen and Lyubich) obtained from the central renormalization of $\hat{P}$ around the super attracting basin $\hat{\Lambda}$. Here $\Lambda'$ is the hole coming from the non-fixed pre-image $\hat{\Lambda}'$ of the super attracting basin $\hat{\Lambda}$ for $\hat{P}$. 
Illustration.
Illustrations.
Proof strategy II.

2 Change the complex structure on $\Lambda'$ and its pre-images under iteration by $f$ if necessary. The need for and possibility to do this will be described below. Denote also by $f$ the new map.

3 Restrict to a disk bounded by some appropriate equipotential level. And perform a more or less standard cut and replace surgery on the sub-wake $W'_k$, $W_k \supset W'_k \supset \Lambda'$ for $f$ to obtain a cubic polynomial like map with a central renormalization hybridly equivalent to $Q_c$ and a peripheral renormalization, a $q$-renormalization around the second critical point which is hybridly equivalent to $Q_d$. 
Proof strategy III.

4. Apply the usual straightening theorem to obtain the desired cubic polynomial $P = P_{a,b}$ by sending the central critical point to 0 and normalize at $\infty$ so that $P$ is monic and $R_P(0)$ lands at the $\beta$-fixed point of the central renormalization.

**Theorem**

*Cubic polynomials $P_{a,b}$ satisfying the specifications of the theorem are q-c-rigid.*

5. Thus the so constructed polynomial $P$ is uniquely defined up to affine conjugacy by the specifications of the theorem.
Illustrations.
Illustrations.
Illustrations.