THE CUBIC CONNECTEDNEDSS LOCUS: LEMON LIMBS

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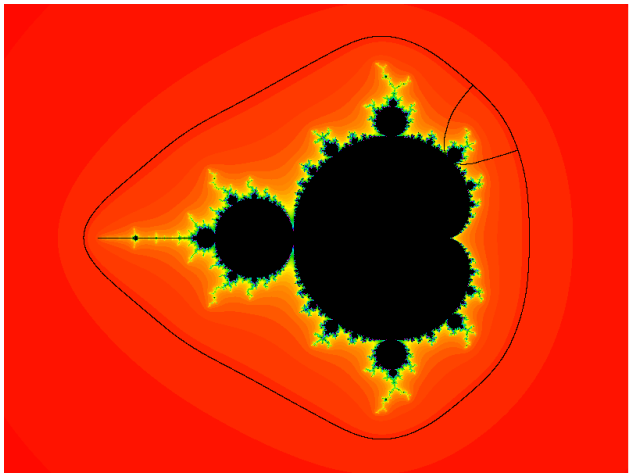
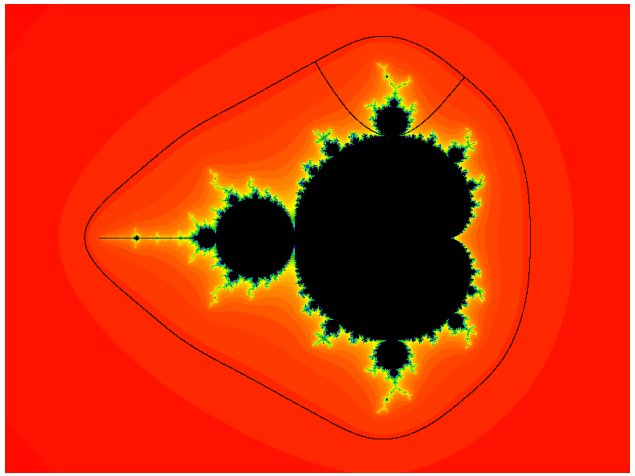
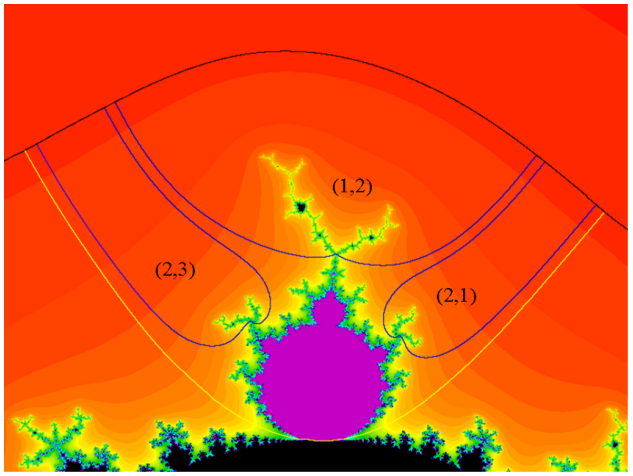
GUIDING QUESTION.

- Limb decomposition has proven to be a very effective way to analyse the Mandelbrot set.
- Is there a fruitfull similar divide and conqour approach to understanding the cubic connectedness locus $\mathcal{C}$?
- Does the cubic connectedness locus also have a tangible limb structure?







Introduction.

We will be working with cubic polynomials in the normal form

$$P_{a,b} : z \mapsto z^3 + 3az^2 + b \qquad a, b \in \mathbb{C} \tag{1}$$

with marked critical points at $0, -2a$. More precisely, we have

critical points	0	$-2a$
critical values	b	$4a^3 + b$
co-critical points	$-3a$	a

Basic notation I.

- We will think of 0 and $-2a$ as the "first" and "second" critical points, respectively.
- The space $\mathcal{P}(3)$ of all such critically marked cubics is isomorphic to $\mathbb{C}^2$ with coordinates (a, b).
- Two distinct cubics $P_{a,b}$ and $P_{a',b'}$ are conjugate by an affine map preserving the marking if and only if $a' = -a, b' = -b$. The unique such conjugacy is $\tau(z) = -z$.
- Thus, the space of all affine conjugacy classes (the so-called "moduli space") of marked cubics is also isomorphic to $\mathbb{C}^2$ with coordinates (a^2, b^2).
- Note however that conjugacy by τ changes the labelling of external rays, i.e it shifts all arguments by $\frac{1}{2}$ mod 1.

Basic notation II.

- $Q_c : z \mapsto z^2 + c$, with Julia set and filled Julia set J_c and K_c, Böttcher coordinate ϕ_c, external rays $\mathcal{R}_c(\theta)$
- M: the Mandelbrot set, with external rays $\mathcal{R}_{\mathrm{M}}(\theta)$
- $P_{a,b}(z) = z^3 + 3az^2 + b$, with marked critical points at 0 and $-2a$, filled Julia set and Julia set $K_{a,b}$ and $J_{a,b}$, Böttcher coordinate $\phi_{a,b}$, external rays $\mathcal{R}_P(\theta)$
- $\mathcal{P}(3) \cong \mathbb{C}^2, \mathcal{C}(3), \mathcal{E}(3)$: The cubic parameter space, connected locus and escape locus
- $\mathcal{H}$: The principal hyperbolic component in $\mathcal{C}(3)$ containing $P_{0,0} : z \mapsto z^3$.

The Lemon family – The starting point –.

- We call the one complex dimensional family

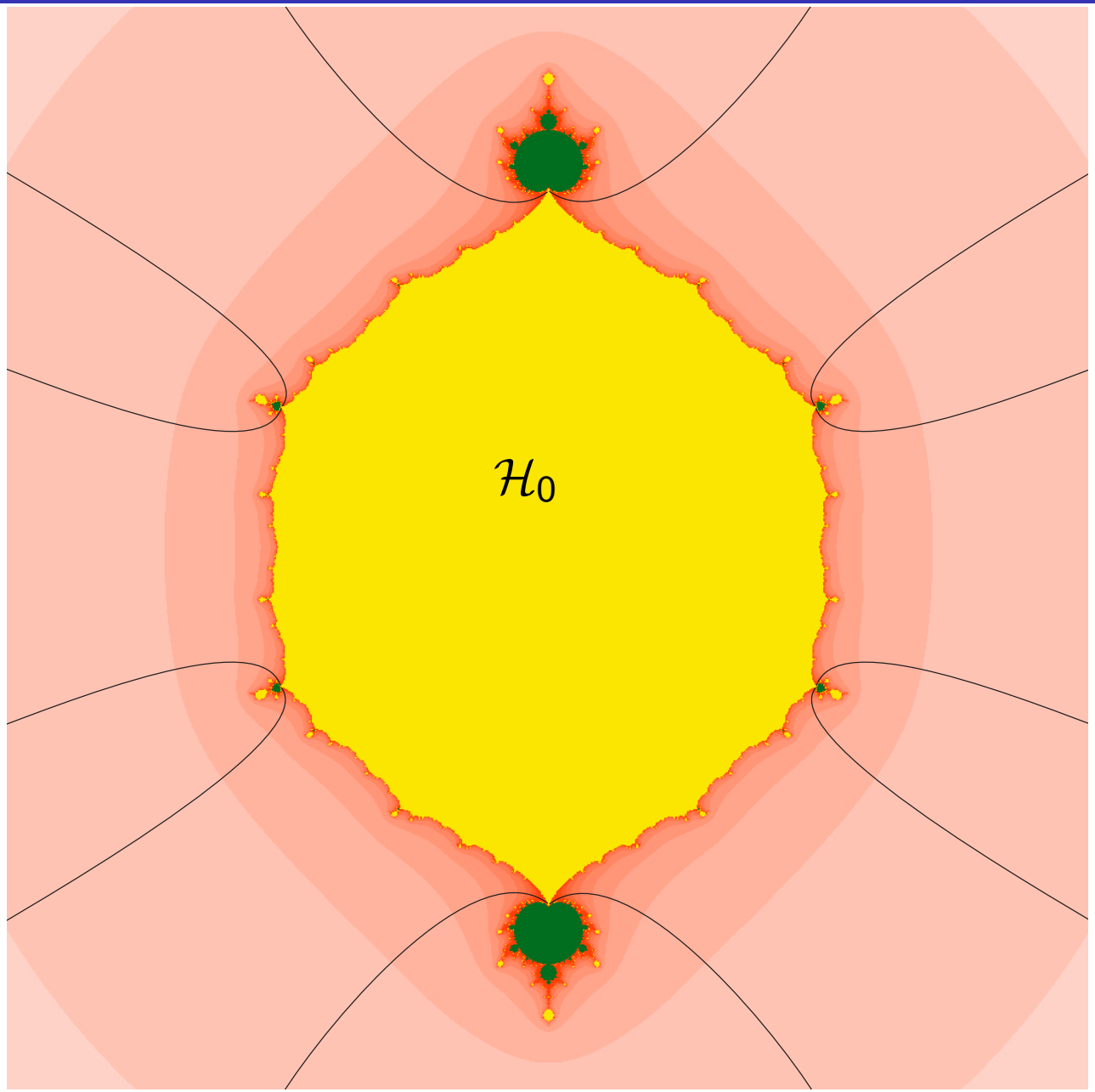
$$P_a(z) = P_{a,0} = z^3 + 3az^2, \qquad a \in \mathbb{C}$$

  the *Lemon Family*.
- The point 0 is a persistent super attracting fixed point in the Lemon family. We denote by Λ_a the immediate basin of 0 for P_a.
- The central hyperbolic component $\mathcal{H}_0$ of the Lemon family is

$$\mathcal{H}_0 := \{a \mid -2a \in \Lambda_a\}$$

- By the Faught-Roesch theorem
  1. the immediate basin Λ_a is a Jordan domain, when $a \notin \mathcal{H}_0$.
  2. $\mathcal{H}_0$ is a Jordan domain and P_a has a quadratic-like restriction hybridly equivalent to Q_0 and with filled-in Julia set $\overline{\Lambda}_a$ if and only if $a \notin \overline{\mathcal{H}}_0$.

The Lemon $\mathcal{H}_0$ and some of its limbs.



Central renormalization

Definition

We say that $P = P_{a,b} \in \mathcal{P}(3)$ is **centrally renormalizable** if P has a quadratic-like restriction $P : U \to U'$ with $0 \in U$ and with connected filled Julia set. Thus $P^{\circ n}(0) \in U$ for all n while $-2a \notin U$.

The central renormalization curves

Definition

Let $c \in \mathcal{M}$ and define $\mathcal{X}_c \subset \mathcal{P}(3)$ as the set of all centrally renormalizable $P_{a,b}$, whose quadratic-like restriction around 0 is hybrid equivalent to Q_c.

- By definition, the sets $\mathcal{X}_c$ for distinct c are disjoint.
- $\mathcal{X}_0 = \mathbb{C} \setminus \overline{\mathcal{H}}_0$

Lemma

$\mathcal{X}_c$ is a one-dimensional complex analytic set for every $c \in \mathrm{M}$, that is, every point in $\mathcal{X}_c$ has an open neighborhood $U \subset \mathcal{P}(3)$ such that $\mathcal{X}_c \cap U$ is biholomorphic to the disk $\mathbb{D}$.

The escape region and $\mathcal{X}_c$.

Definition

We denote by $\mathcal{E}_c$ the set of parameters in $\mathcal{X}_c$ for which the second critical point $-2a$ escapes to ∞

$$\mathcal{E}_c = \mathcal{X}_c \cap \mathcal{E}(3).$$

Theorem (Branner-Hubbard)

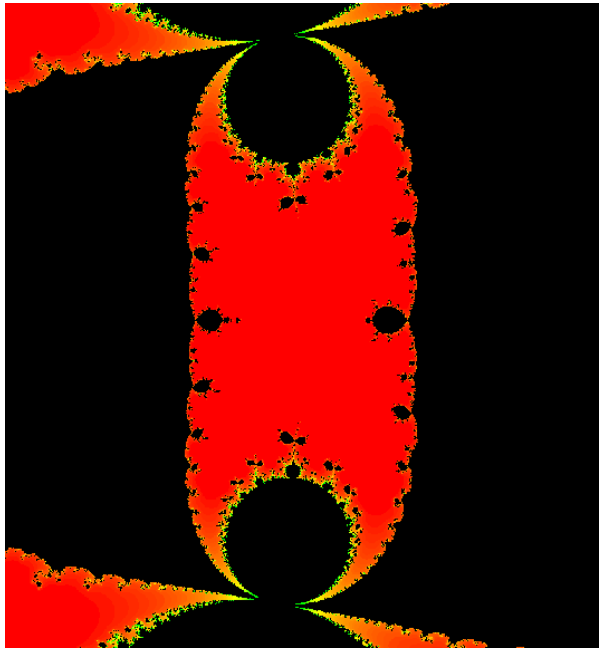
The set $\mathcal{E}_c$ is isomorphic to the punctured disk $\mathbb{C} \setminus \overline{\mathbb{D}}$.

$\mathcal{E}_c$ will henceforth be denoted the escape region in $\mathcal{X}_c$.
All cubics in $\mathcal{E}_c$ are quasiconformally conjugate.

Question

Is $\mathcal{X}_c$ connected? Is it embedded in $\mathcal{P}(3)$? Is $\pi_1(\mathcal{X}_c)$ infinite cyclic generated by the loop around the puncture in $\mathcal{E}_c$?

Examples of $\mathcal{E}_c$.



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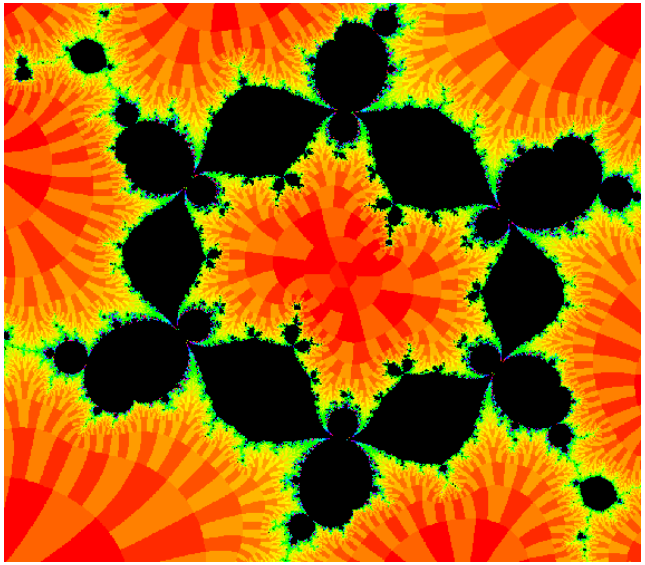

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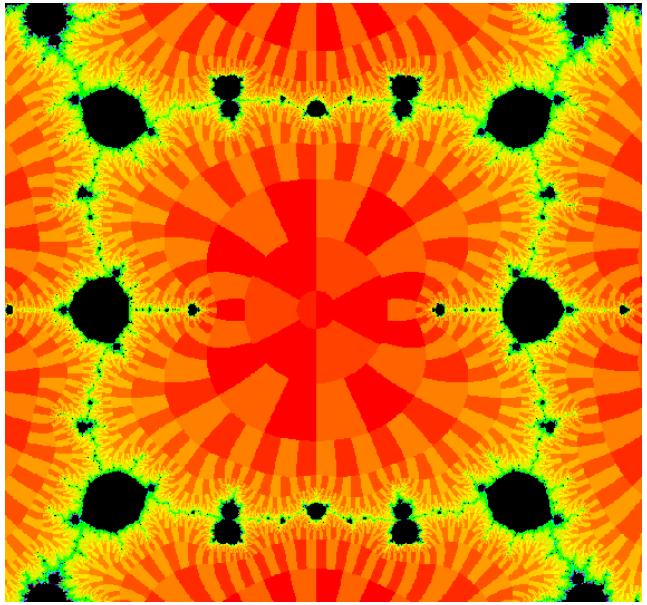

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Combinatorics of periodic orbits.

- Let $f : \mathbb{T} \to \mathbb{T} := \mathbb{R}/\mathbb{Z}$ be a degree $k \geq 2$ covering map.
- Example : the multiplication-by-k map $\mathbf{m}_k$: $t \mapsto kt \pmod{\mathbb{Z}}$.
- Let $\mathcal{O} = \{t_1, \ldots, t_q\}$ with $0 \leq t_1 < \cdots < t_q < 1$ be a q-periodic orbit for f.
- Example $f = \mathbf{m}_2$ and $\mathcal{O} = \{1/5, 2/5, 3/5, 4/5\}$ with $k = 2$ and $q = 4$.
- By the **combinatorics** of $\mathcal{O}$ under f we mean the cyclic permutation σ of $\{1, \ldots, q\}$ defined by

$$f(t_j) = t_{\sigma(j)} \qquad \text{for all } 1 \leq j \leq q.$$

- in the example above $\sigma = (1, 2, 4, 3)$.

Illustrations.

Orbits enforcing

Theorem

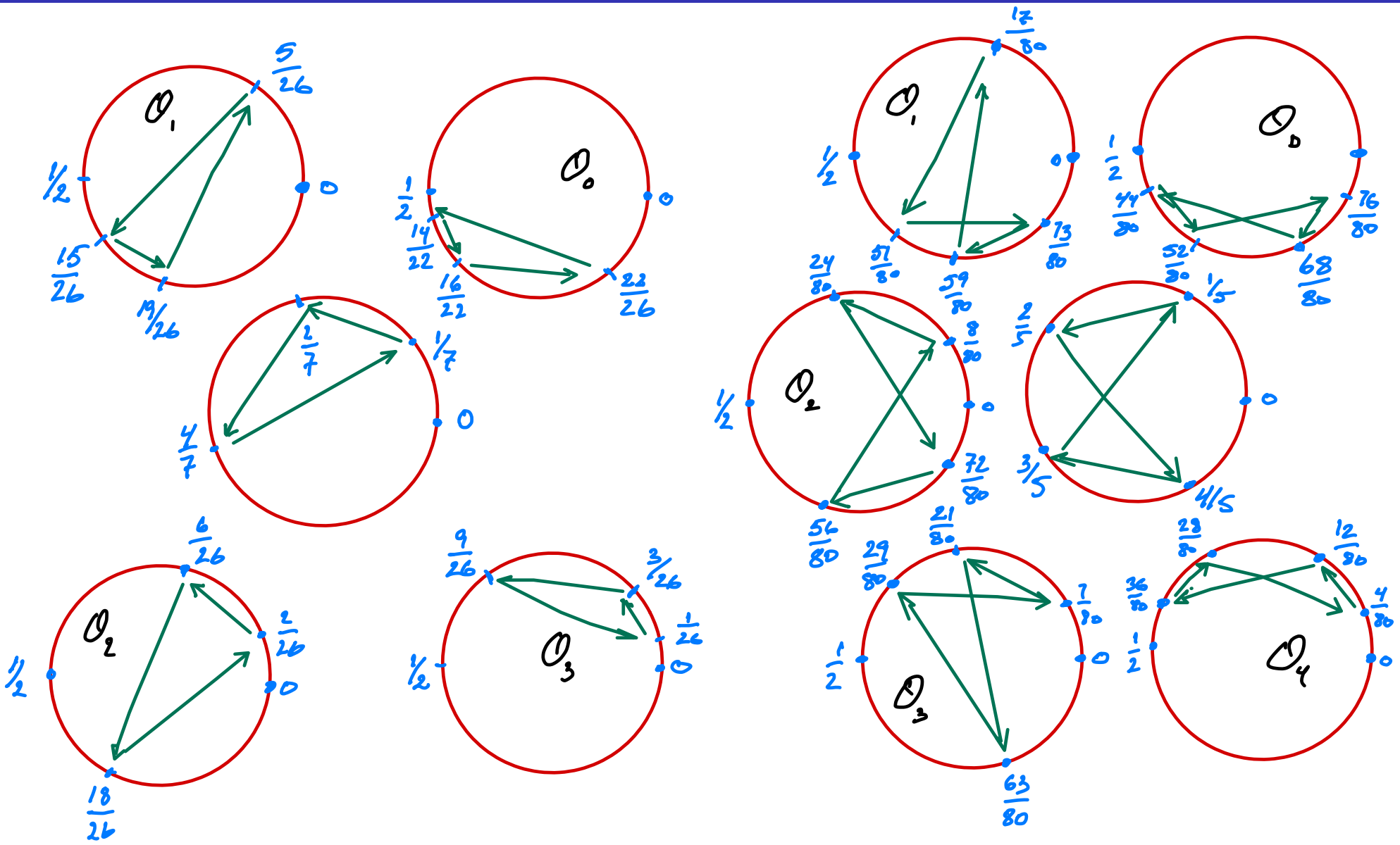
A given combinatorics σ is realised at most once by $\mathbf{m}_2$.

Theorem

Every q-periodic combinatorics σ realised by $\mathbf{m}_2$ is realised precisely $q+1$ times under $\mathbf{m}_3$ by orbits $\mathcal{O}_0, \ldots, \mathcal{O}_q$. For $q > 1$ these orbits are uniquely determined by the deployment condition

$$\#(\mathcal{O}_j \cap [0, 1/2[\,) = j \qquad 0 \leq j \leq q. \tag{2}$$

Illustrations.



Orbit interlacing and simulating orbits

Theorem

Let $\{t_1, \ldots, t_q\}$ *be an* $\mathbf{m}_2$ *orbit of period* $q \geq 2$ *and combinatorics* σ*. For each* $1 \leq k \leq q$*, the neighboring orbits* $\mathcal{O}_{k-1}, \mathcal{O}_k$ *interlace. More precisely, if* $\mathcal{O}_k = \{x_1, \ldots, x_q\}$ *and* $\mathcal{O}_{k-1} = \{y_1, \ldots, y_q\}$*, then*

$$0 < x_1 < y_1 < \cdots < x_k < \frac{1}{2} < y_k < \cdots < x_q < y_q < 1. \tag{3}$$

Definition

We call the pair

$$\mathcal{O}^-(t) := \mathcal{O}_k \quad \text{and} \quad \mathcal{O}^+(t) := \mathcal{O}_{k-1}$$

the **simulating orbits** for the periodic point $t = t_k$.
We extend the above definitions to the fixed point $t = 0$ by setting

$$\mathcal{O}^-(0) := \{0\} \quad \text{and} \quad \mathcal{O}^+(0) := \{1/2\}.$$

Simulating orbits.

- We call the pair points

$$x_k \in \mathcal{O}^-(t) \qquad y_k \in \mathcal{O}^+(t)$$

the **critical angles** associated with t and

- $$\mathbf{m}_3(x_k) = x_{\sigma(k)} \qquad \mathbf{m}_3(y_k) = y_{\sigma(k)}$$

the **critical value angles** associated with t.

- An easy computation shows that

$$y_k - x_k = \frac{3^{q-1}}{3^q - 1} \quad \text{and} \quad y_{\sigma(k)} - x_{\sigma(k)} = \frac{1}{3^q - 1}.$$

rabbit

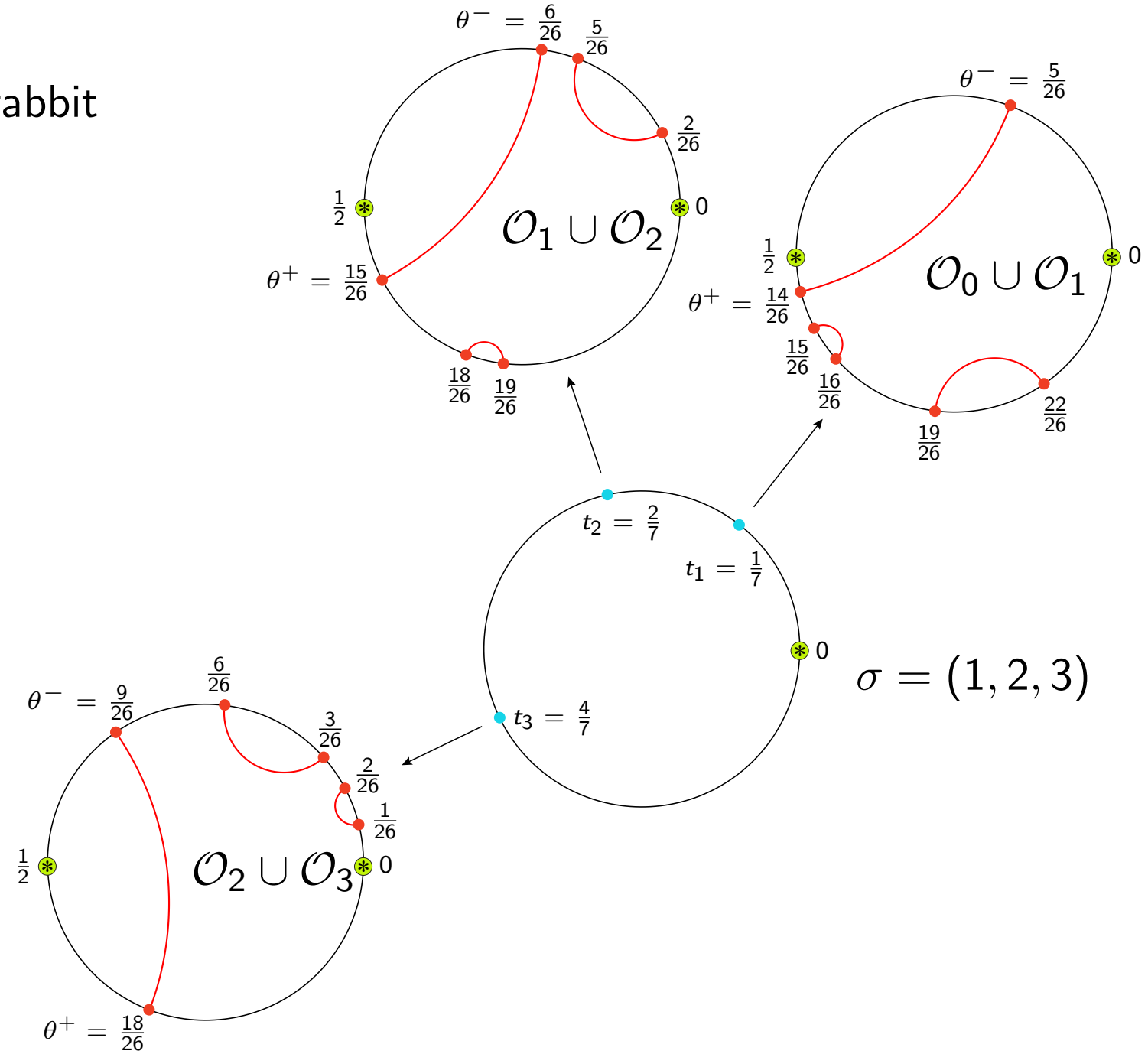
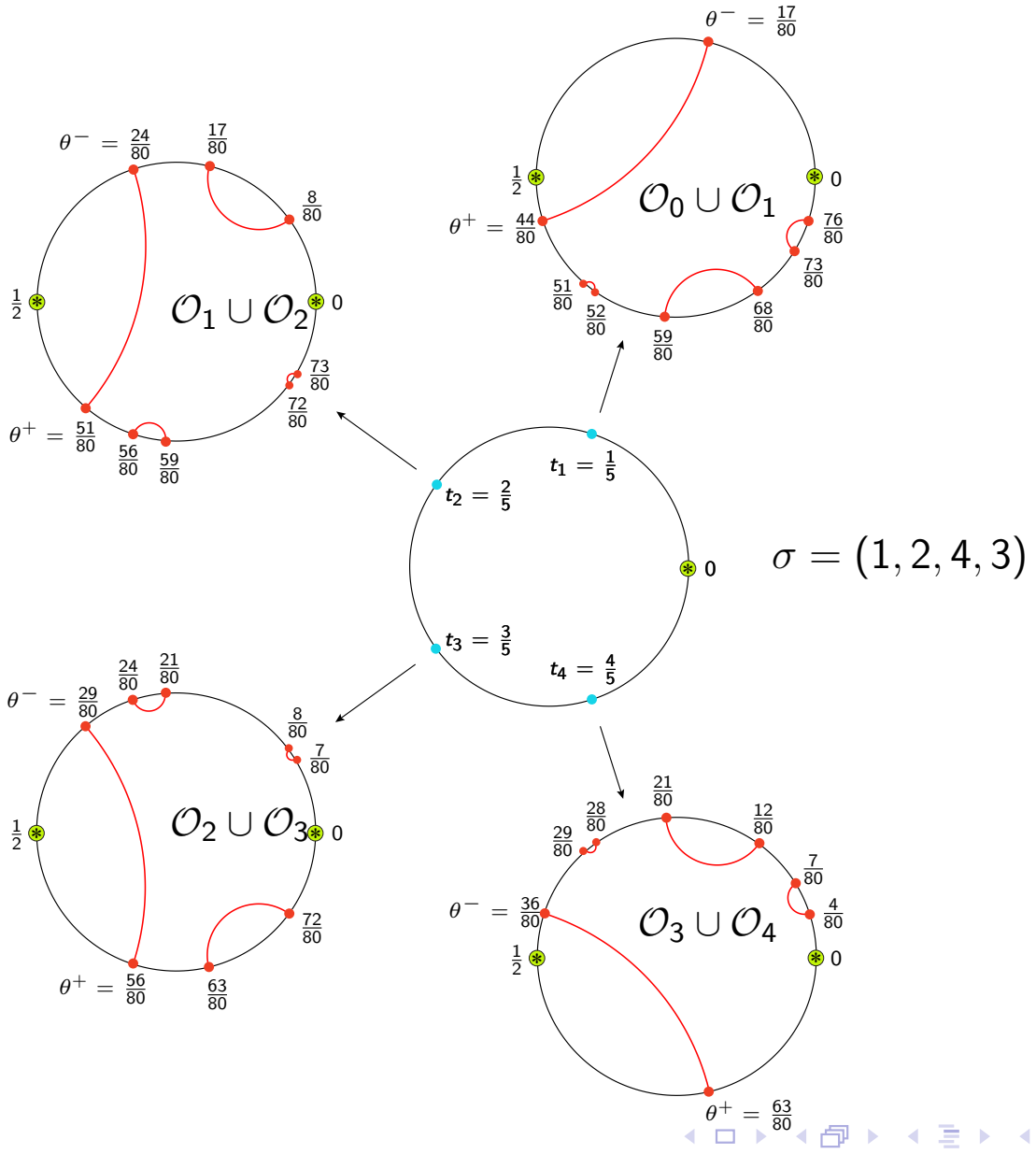

$\theta^- = \frac{6}{26}$
$\frac{5}{26}$
$\frac{2}{26}$
$\frac{1}{2}$
0
$\mathcal{O}_1 \cup \mathcal{O}_2$
$\theta^+ = \frac{15}{26}$
$\frac{18}{26}$
$\frac{19}{26}$
$\theta^- = \frac{5}{26}$
$\frac{1}{2}$
0
$\mathcal{O}_0 \cup \mathcal{O}_1$
$\theta^+ = \frac{14}{26}$
$\frac{15}{26}$
$\frac{16}{26}$
$\frac{19}{26}$
$\frac{22}{26}$
$t_2 = \frac{2}{7}$
$t_1 = \frac{1}{7}$
0
$t_3 = \frac{4}{7}$
$\sigma = (1, 2, 3)$
$\theta^- = \frac{9}{26}$
$\frac{6}{26}$
$\frac{3}{26}$
$\frac{2}{26}$
$\frac{1}{26}$
$\frac{1}{2}$
$\mathcal{O}_2 \cup \mathcal{O}_3$
0
$\theta^+ = \frac{18}{26}$

Illustration.

Cocapelli



Lemon Limbs and Wakes I.

Definition

Let $t = t_k$ be the k-th point in a q-cycle for $\mathbf{m}_2$ as above and let $\mathcal{O}_{k-1} = \{x_1, \ldots, x_q\}$ and $\mathcal{O}_k = \{y_1, \ldots, y_q\}$ be the associated simulating cycles for $\mathbf{m}_3$.
We define the t-**wake** $\mathcal{W}(t) \subset \mathcal{P}(3)$ as

$$\mathcal{W}(t) := \{P \in \mathcal{P}(3) : R_P(x_j) \text{ and } R_P(y_j) \text{ co-land for } j = 1, \ldots q\}$$

and the corresponding **"Lemon"** t-**limb**

$$\mathcal{L}(t) = \mathcal{C}(3) \cap \mathcal{W}(t)$$

Note that the points $z_j(P) = \overline{R_P(x_j)} \cap \overline{R_P(y_j)}$ form a periodic orbit whose period is a divisor of q and that either this orbit is repelling or $(P^q)'(z_j(P)) = 1$. Also note that wakes are neither open nor closed.

Lemon Limbs and Wakes II.

The involution $\theta \mapsto \theta^* := \theta + \frac{1}{2}$ is an automorphism of $\mathbf{m}_3$ exchanging the fixed points 0 and $\frac{1}{2}$ for $\mathbf{m}_3$. We define

$$\mathcal{W}^*(t) := \{P \in \mathcal{P}(3) : R_P(x_j^*) \text{ and } R_P(y_j^*) \text{ co-land for } j = 1, \ldots q\}$$

and

$$\mathcal{L}^*(t) = \mathcal{C}(3) \cap \mathcal{W}^*(t_k)$$

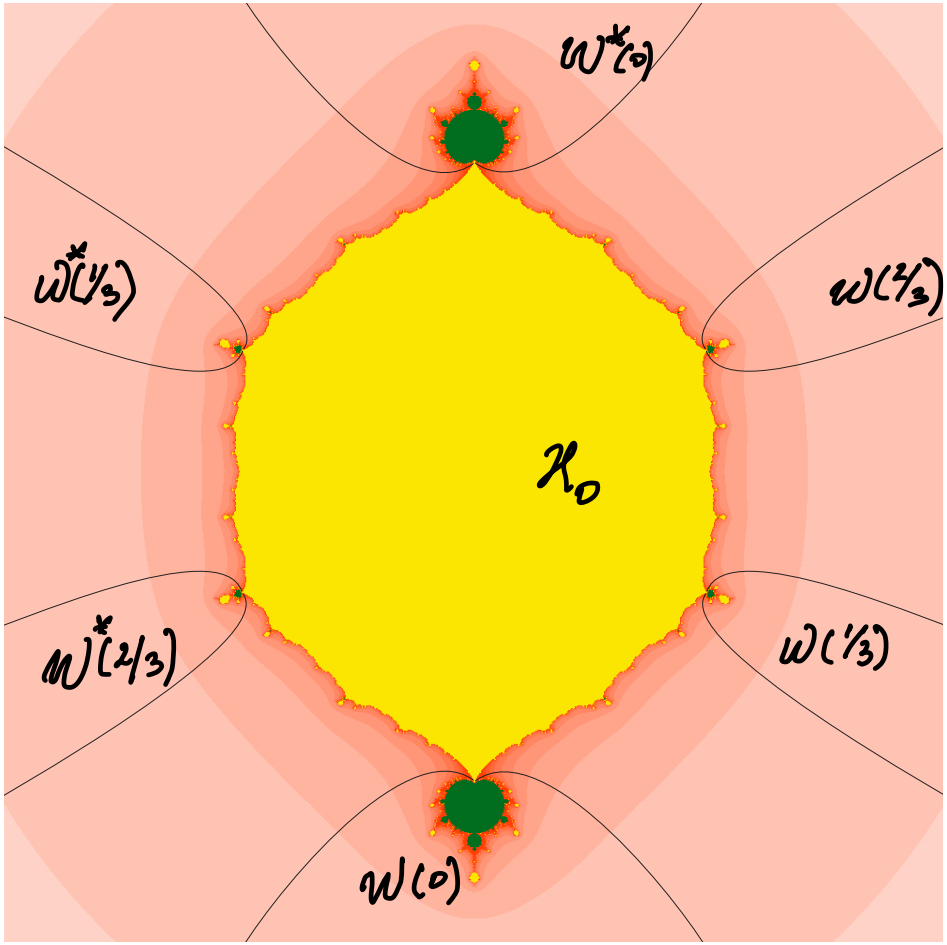
Then P belongs to $\mathcal{W}(t)$ or $\mathcal{L}(t)$ if and only if $\tau \circ P \circ \tau$ belongs to $\mathcal{W}^*(t)$ or $\mathcal{L}^*(t)$ respectively (recall $\tau(z) = -z$).

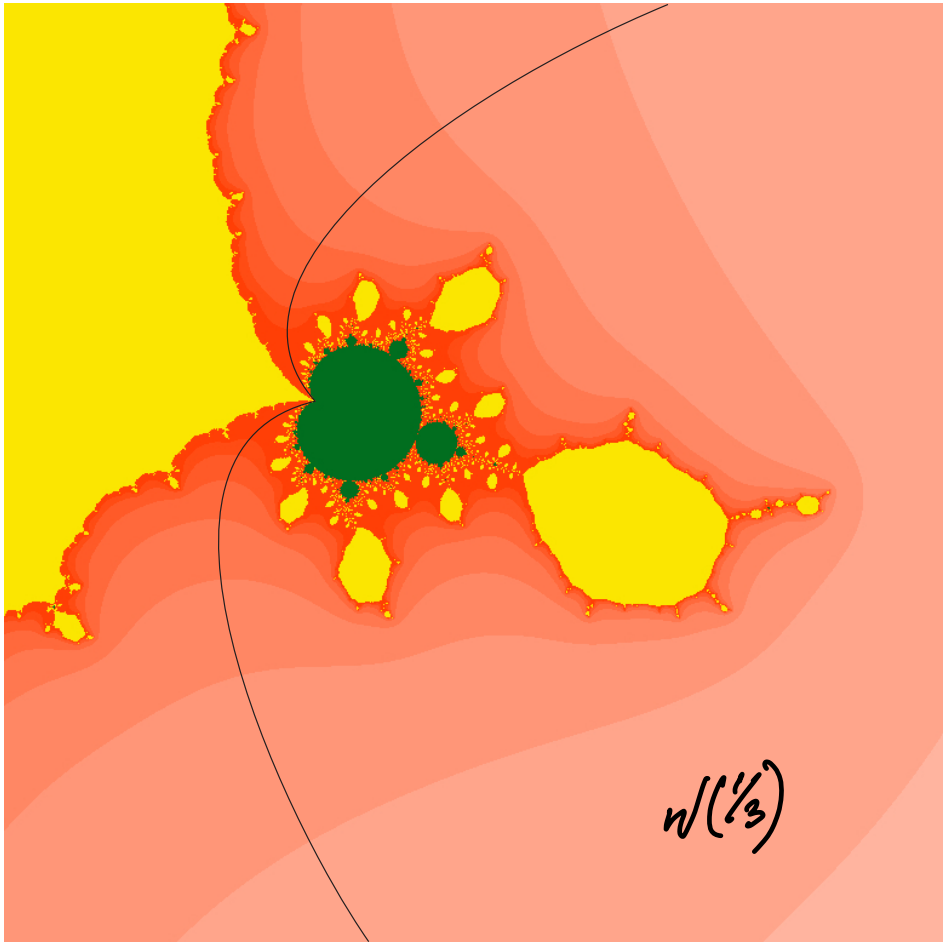
Definition

For $c \in \mathrm{M}$ denote by $\mathcal{L}_c(t)$ and $\mathcal{L}_c^*(t)$ the intersections

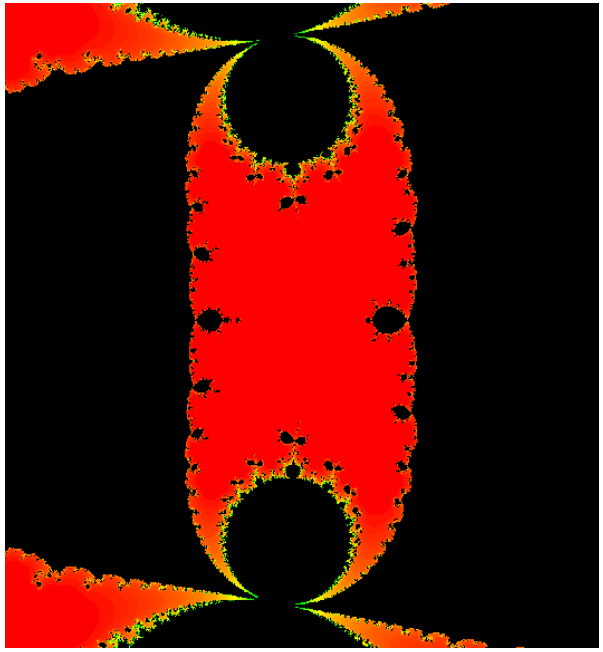
$$\mathcal{L}_c(t) := \overline{\mathcal{X}}_c \cap \mathcal{L}(t) \qquad \mathcal{L}_c^*(t) := \overline{\mathcal{X}}_c \cap \mathcal{L}^*(t).$$

Wakes and Limbs in $\mathcal{X}_0$.





More $\mathcal{E}_c$'s and corresponding conjectural $\mathcal{X}_c$'s.



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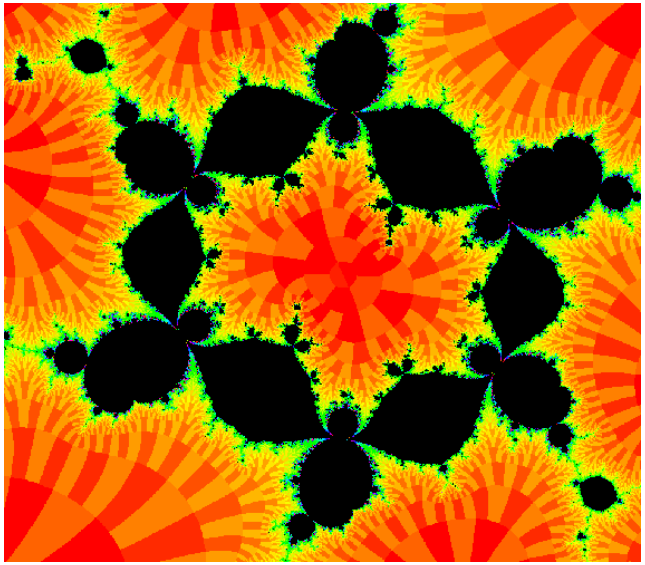

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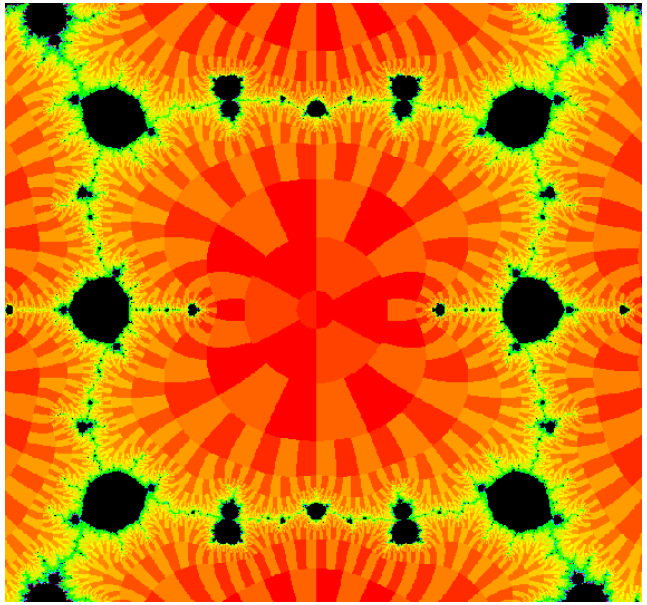

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GUIDING QUESTIONS revisited

- What can we say about the Lemon t-limb for t a periodic point for $\mathbf{m}_2$?

- What can we say about the intersection $\mathcal{L}_c(t)$ of the Lemon t-limb with $\mathcal{X}_c$ for t a periodic point for $\mathbf{m}_2$?
- Is

$$\mathcal{X}_c = \mathcal{E}_c \bigcup_{t,t^* \text{ periodic for } \mathbf{m}_2} \mathcal{L}_c(t) \cup \mathcal{L}_c(t^*) \quad ?$$

- We know from the Faught-Roesch Theorem that for $c = 0$ the answer to this question is yes.

Essential disjointness.

Proposition

Let $0 \leq t \leq s < 1$ *be two periodic points for* $\mathbf{m}_2$. *Then*

1. $$\mathcal{W}(t) \cap \mathcal{W}(s) \neq \emptyset \qquad \Longrightarrow \qquad t = s.$$
2. *For* $c \in \mathrm{M}$
$$\mathcal{W}_c(t) \cap \mathcal{W}_c^*(s) \neq \emptyset \qquad \Longrightarrow$$
$t = s = 0$ *or* $t = t_k, s = t_j$ *belongs to the same rotation cycle* $0 < t_1 < \ldots < t_q < 1$, $\sigma = (1, 2, \ldots, q)^p$ *where* $(p, q) = 1$ *and* $k + j = q + 1$.

For $c \notin \mathrm{M}$ the intersection possibilities of $\mathcal{W}(t)$ and $\mathcal{W}^*(s)$ is much richer.

Basic Limb dynamics

- Let $t = t_k$ belong to a q-cycle $0 < t_1 < \ldots < t_q < 1$ for $\mathbf{m}_2$ with combinatorics σ, let $c \in \mathrm{M}$ and let $P = P_{a,b} \in \mathcal{L}_c(t)$.
- It is not difficult to see that the second critical point $C = -2a$ belongs to the dynamical wake $\mathcal{W}_P(t_k)$ with boundary rays $\mathcal{R}_P(x_k)$ and $\mathcal{R}_P(y_k)$.
- and more generally for each $j = 1 \ldots q$ the iterate $P^j(C)$ belongs to the wake $\mathcal{W}_P(t_{\sigma^j(k)})$ with boundary rays $\mathcal{R}_P(x_{\sigma^j(k)})$ and $\mathcal{R}_P(y_{\sigma^j(k)})$.

Illustrations.

Peripheral renormalizations

Definition

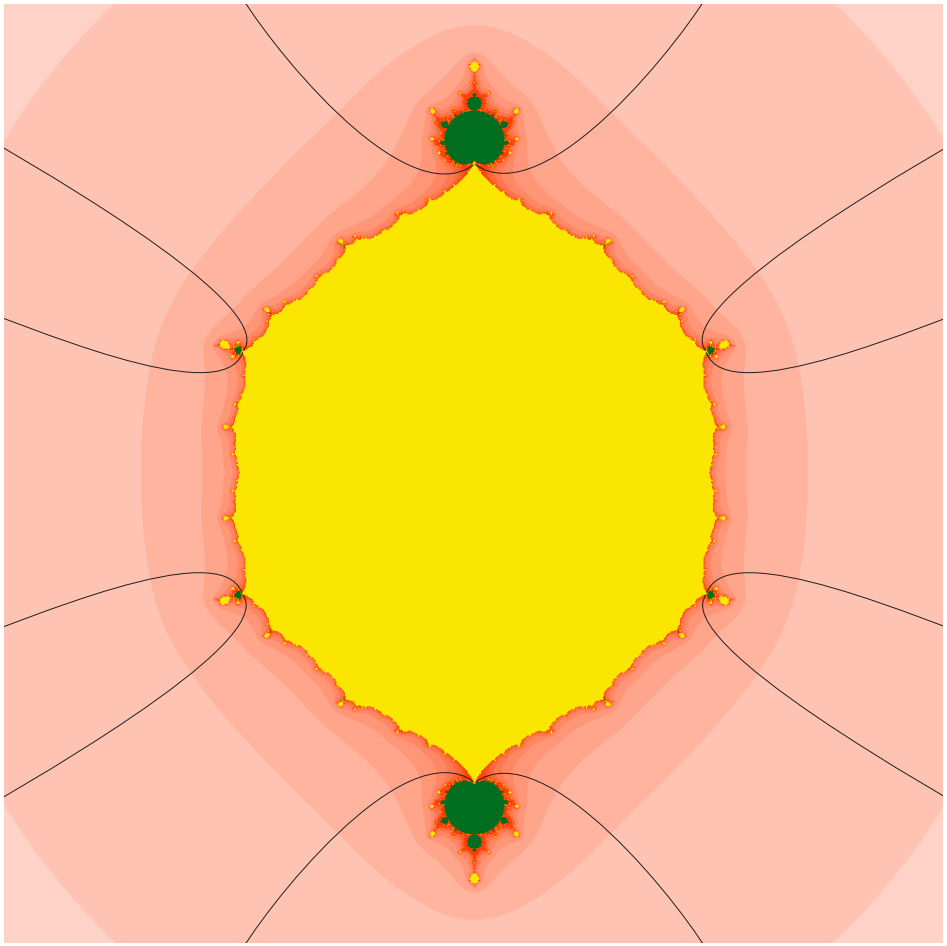
We say that $P \in \mathcal{L}_c(t)$ is **peripherally renormaliseable** iff P^q has a quadratic-like restriction

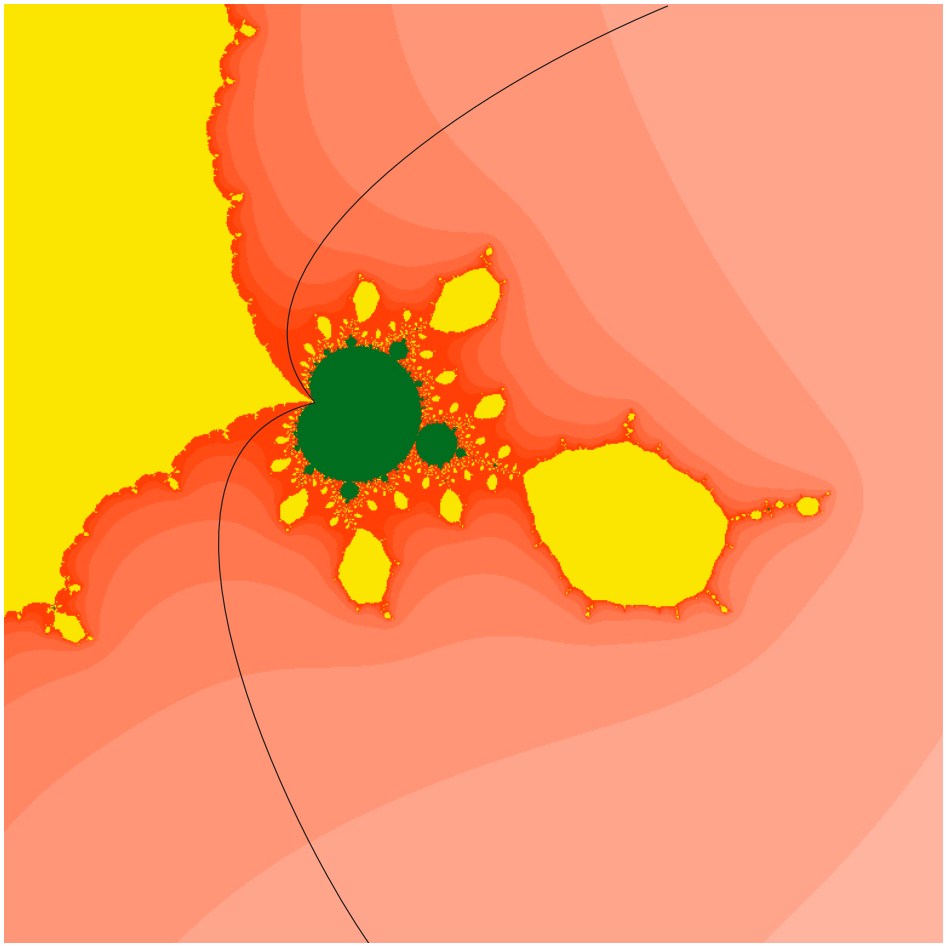
$$P^q : V \to V'$$

with connected filled-in Julia set containing the second or peripheral critical point $C = -2a$.
We call such restriction a **peripheral renormalisation** of P.

Illustration.

Mandelbrot copies corresponding to peripheral renormalization in $\mathcal{X}_0$.

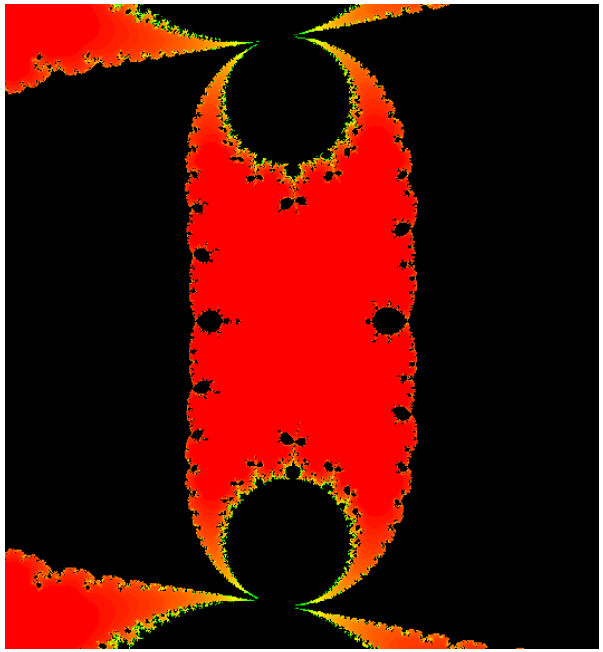


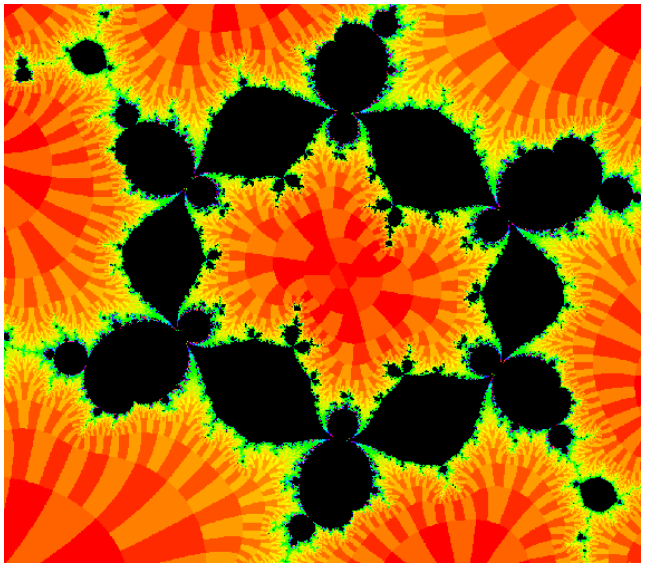


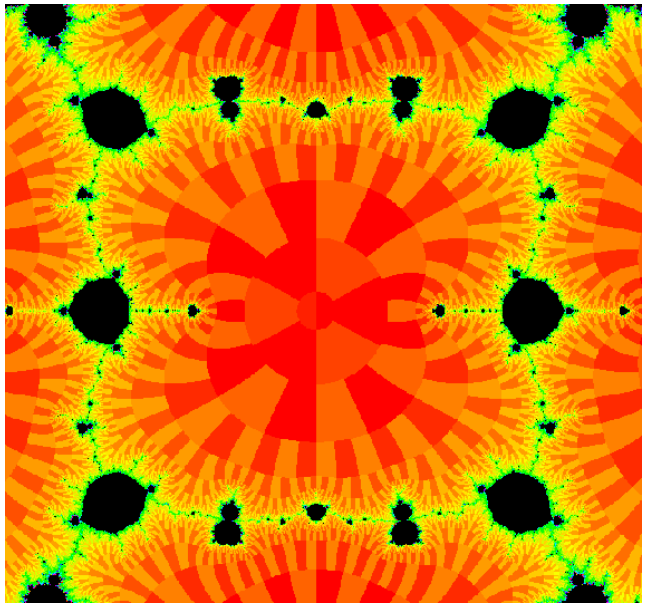
Main Theorem

Theorem

Let t be periodic for $\mathbf{m}_2$ with orbit $0 < t_1 < \ldots t = t_k, \ldots < t_q < 1$. Let $c \in \mathrm{M} \setminus \{\overline{\mathcal{R}}_{\mathrm{M}}(t_1), \ldots, \overline{\mathcal{R}}_{\mathrm{M}}(t_q)\}$ and let $d \in \mathrm{M} \setminus \{1/4\}$. Then there exists a unique cubic polynomial $P = P_{a,b} \in \mathcal{L}_c(t) = \mathcal{X}_c \cap \mathcal{L}(t)$ with peripheral renormalisation $P^q : V \to V'$ hybridly equivalent to Q_d.





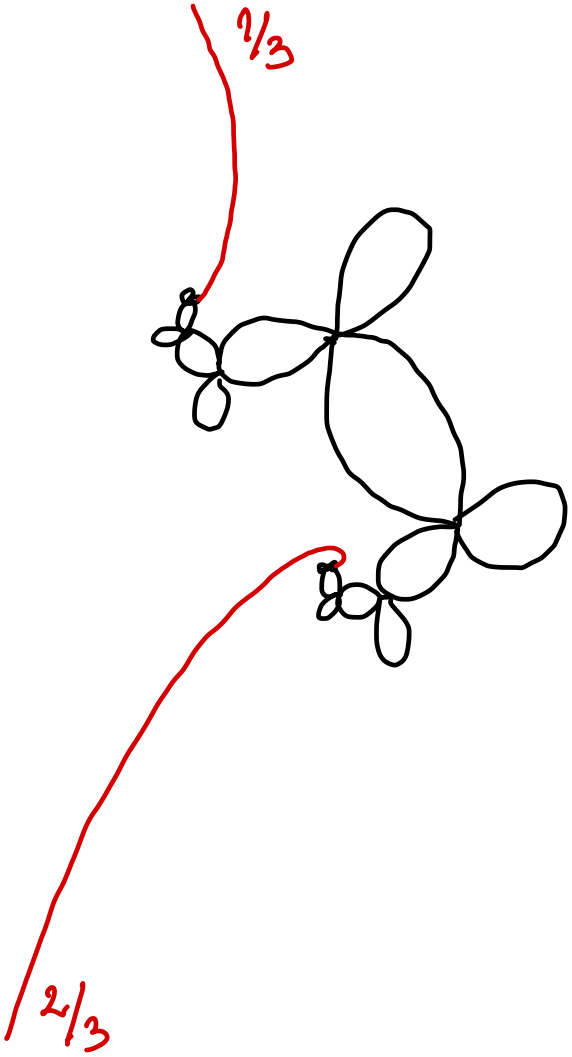


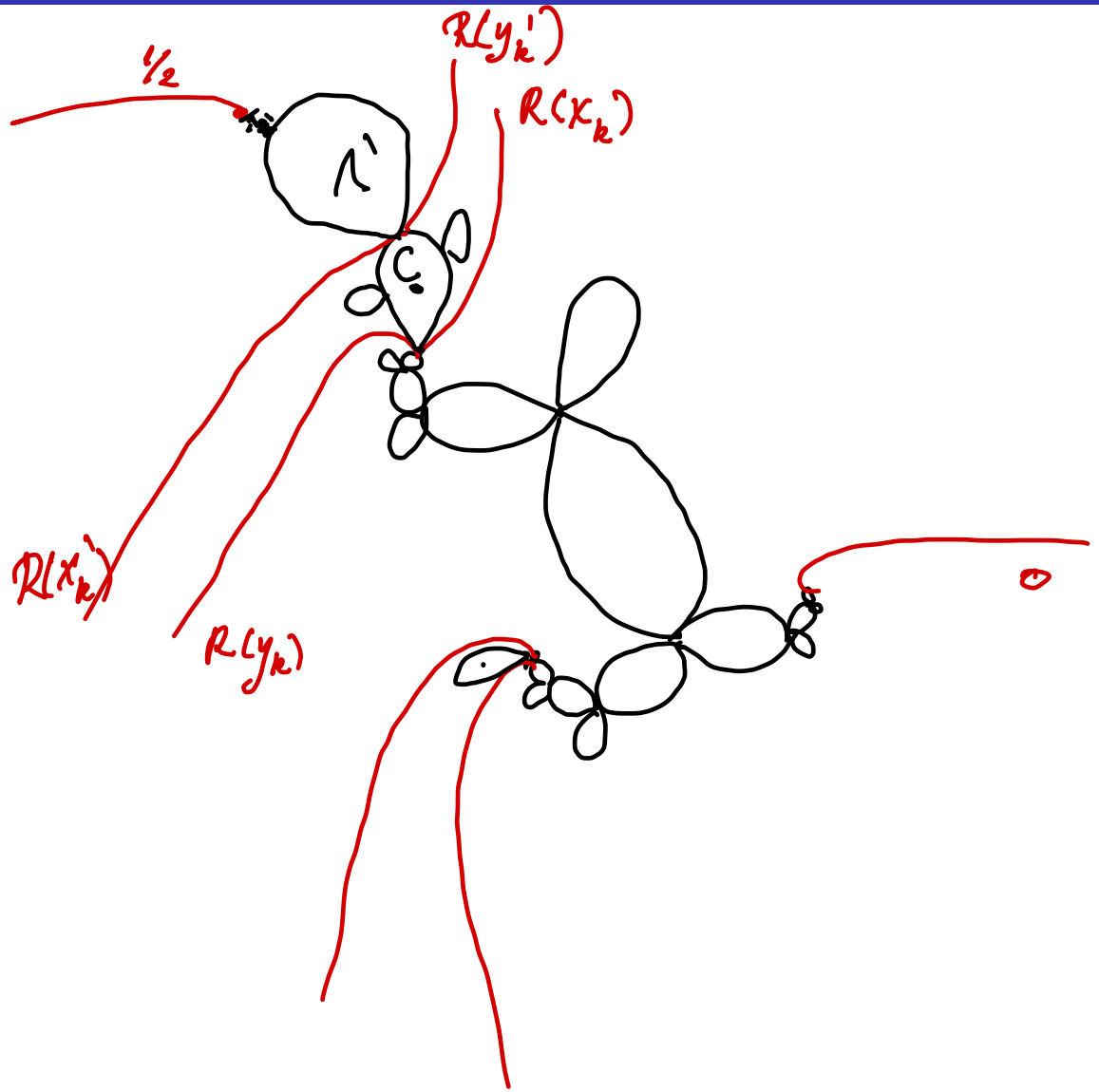
Proof strategy I.

1. Start from the quadratic polynomial Q_c and the unique cubic polynomial $\widehat{\mathrm{P}} = P_a \in \mathrm{Per}_1(0) \cap \mathcal{L}(t_k)$, which is peripherally renormalizeable with peripheral renormalisation hybridly equivalent to Q_d. Construct a new holomorphic map $f : \mathbb{C} \setminus \Lambda' \to \mathbb{C}$ by changing the external class Q_0 of Q_c to the external class of the quadratic-like map (in the sense of McMullen and Lyubich) obtained from the central renormalization of $\widehat{\mathrm{P}}$ around the super attracting basin $\widehat{\Lambda}$. Here Λ' is the hole coming from the non-fixed pre-image $\widehat{\Lambda}'$ of the super attracting basin $\widehat{\Lambda}$ for $\widehat{\mathrm{P}}$.

Illustration.

Illustrations.





Proof strategy II.

2. Change the complex structure on Λ' and its pre-images under iteration by f if necessary. The need for and possibility to do this will be described below. Denote also by f the new map.
3. Restrict to a disk bounded by some appropriate equipotential level. And perform a more or less standard cut and replace surgery on the sub-wake $W_k', W_k \supset W_k' \supset \Lambda'$ for f to obtain a cubic polynomial like map with a central renormalization hybridly equivalent to Q_c and a peripheral renormalization, a q-renormalization around the second critical point which is hybridly equivalent to Q_d.

Illustrations.

Proof strategy III.

4. Apply the usual straightening theorem to obtain the desired cubic polynomial $P = P_{a,b}$ by sending the central critical point to 0 and normalize at ∞ so that P is monic and $R_P(0)$ lands at the β-fixed point of the central renormalization.

Theorem

Cubic polynomials $P_{a,b}$ satisfying the specifications of the theorem are q-c-rigid.

5. Thus the so constructed polynomial P is uniquely defined up to affine conjugacy by the specifications of the theorem.

Illustrations.

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