Polynomials with Bounded Type Siegel Disks

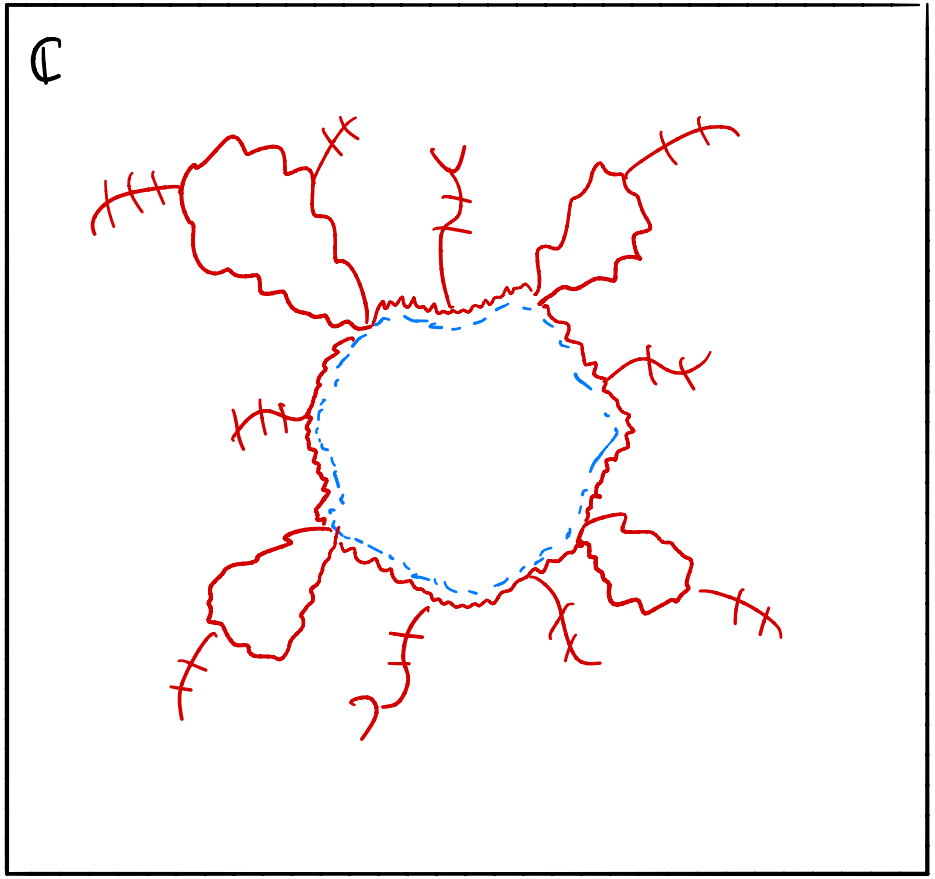
Jonguk Yang, Stony Brook University

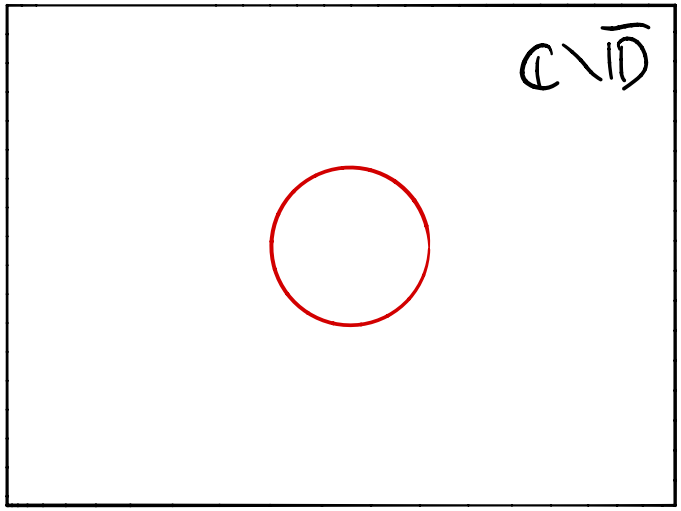
On Geometric Complexity of Julia Sets II, August 2020

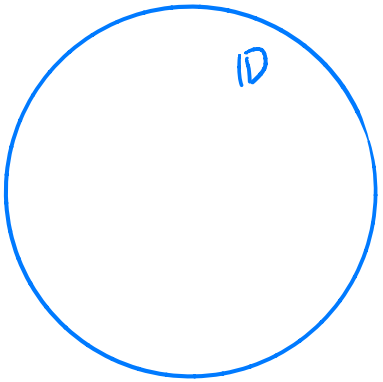
Set Up

Consider a polynomial f of degree $d \geq 2$ that has a Siegel point at 0 with rotation number ρ. Let $0 \in \Delta_f$ be the Siegel disk, A_f^∞ the attracting basin of infinity, K_f the filled Julia set, and J_f the Julia set. WLOG, assume

- 0 is fixed.
- K_f is connected.







Statement of the Theorem

Theorem (Shishikura, Zhang*)

Suppose ρ is of bounded type. Then $\partial\Delta_f$ is a quasicircle that contains at least one critical point. Consequently, ϕ_0 extends homeomorphically to $\partial\Delta_f$.

* Zhang proved the theorem for all rational maps.

Recall that ϕ_∞^{-1} extends continuously to $\partial\mathbb{D} \iff J_f$ is locally connected.

Theorem (Y.)

Suppose ρ is of bounded type. Then J_f is locally connected at every point in $\partial\Delta_f$.

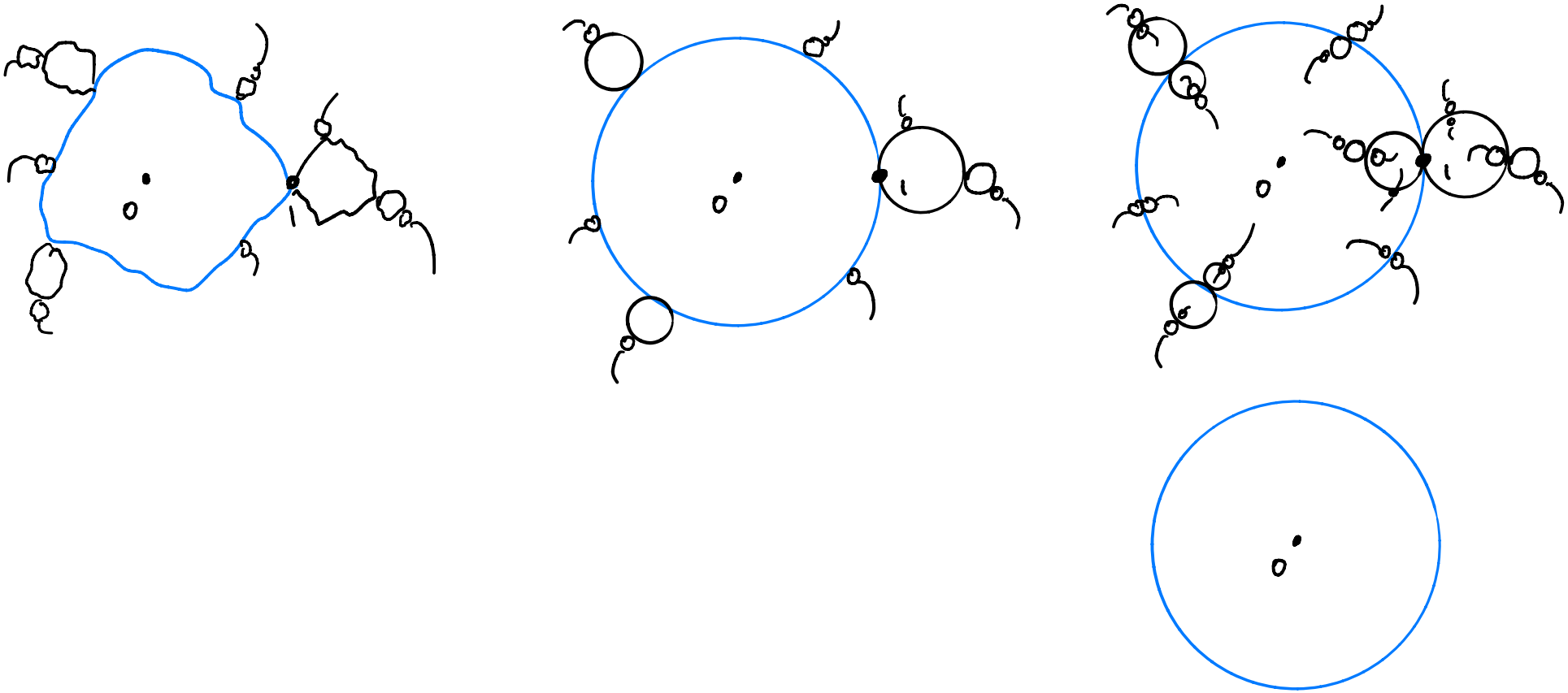
The analogous theorem for attracting components was proved by Kozlovski-van Strien, and for parabolic components by Roesch-Yin.

Blaschke Product Model

Theorem

Suppose ρ is of bounded type. Then there exists a Blaschke product F of degree $2d-1$ such that f is quasiconformally conjugate to the modified Blaschke product $\tilde{F}$ obtained from F via the Douady-Ghys surgery.

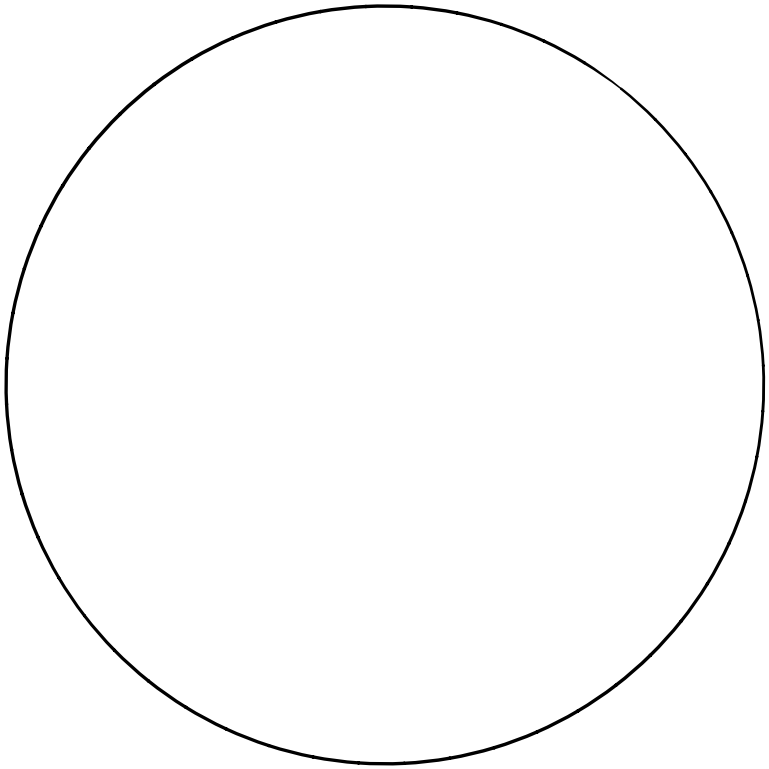
Henceforth, we assume that F has a critical point at 1.

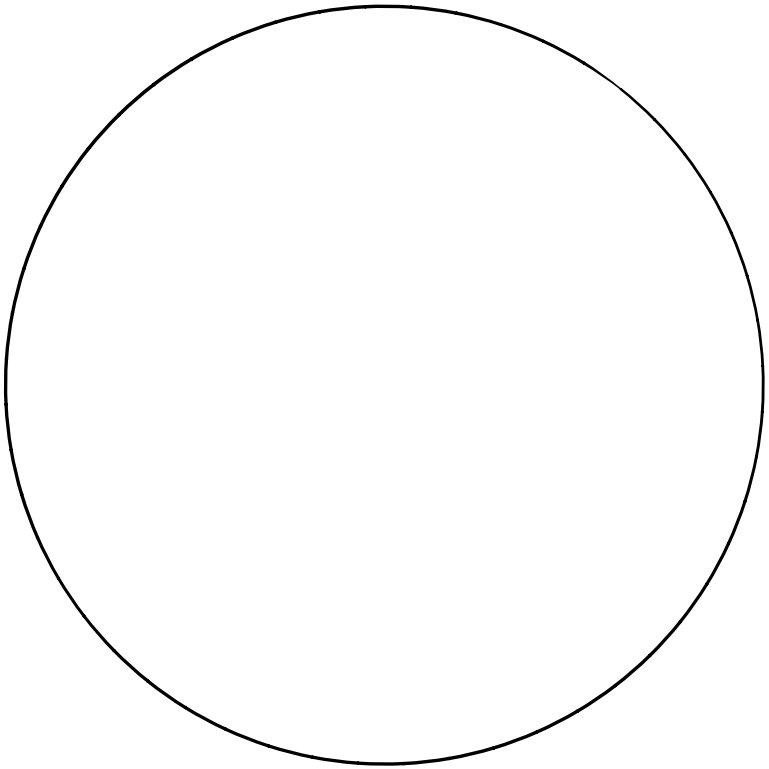


Dynamic Partition of the Circle

Consider an analytic circle homeomorphism F with rotation number ρ. Choose $x_0 \in \partial\mathbb{D}$. Let q_n be the nth closest return time, and let I_n be the nth closest return arc bounded by x_0 and $x_{q_n} = F^{q_n}(x_0)$. The *nth dynamic partition* of $\partial\mathbb{D}$ is defined as

$$\mathcal{I}_n = \{F^i(I_n) \mid 0 \leq i < q_{n+1}\} \cup \{F^i(I_{n+1}) \mid 0 \leq i < q_n\}.$$

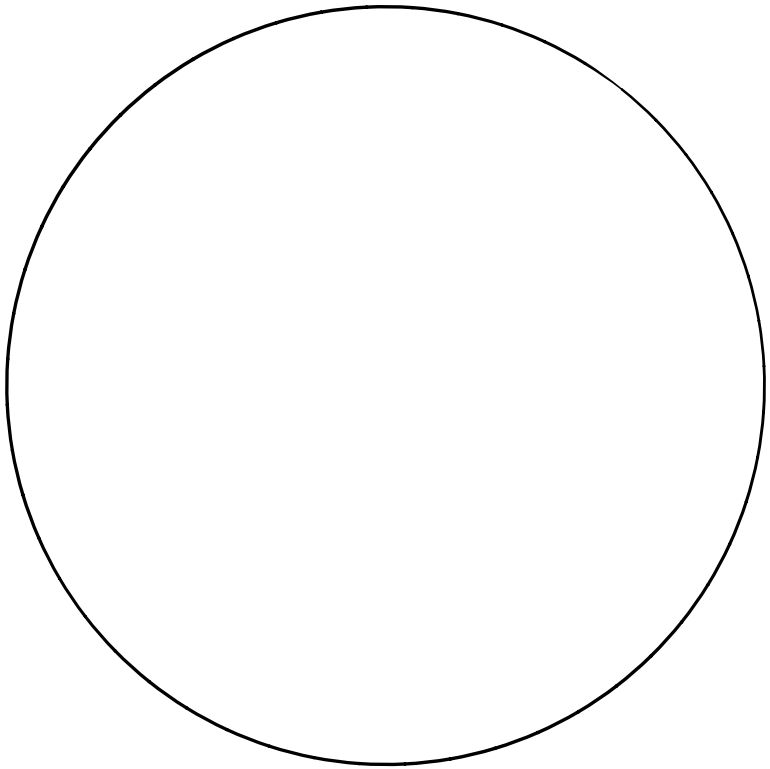




Real A Priori Bounds

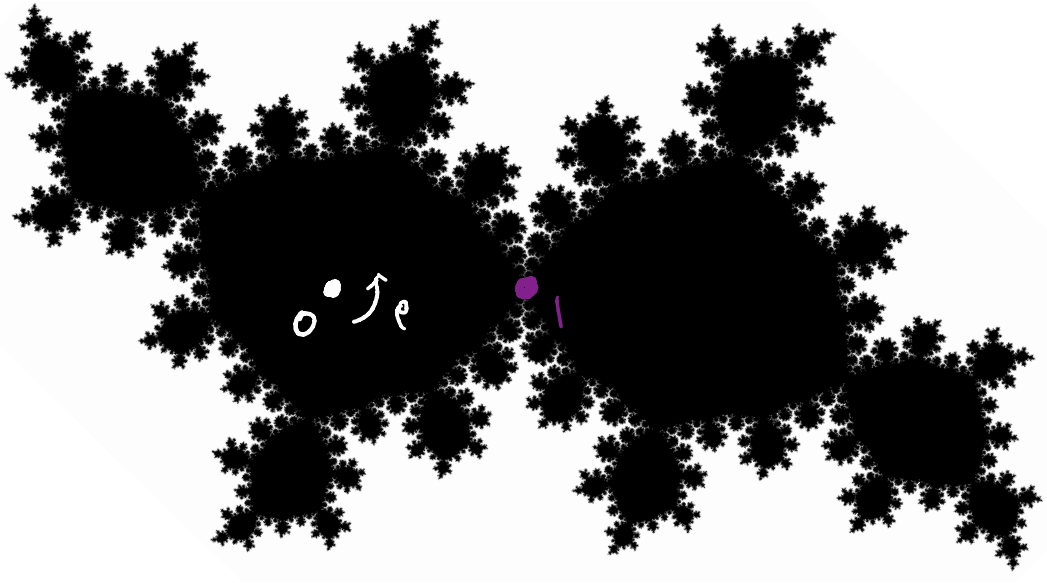
Theorem (Herman)

For all adjacent arcs $I, J \in \mathcal{I}_n$, we have $C^{-1}|J| < |I| < C|J|$ for some uniform $C > 1$. Moreover, there exists a uniform constant $\lambda > 0$ such that $F^{q_{n+1}}$ restricted to the $\lambda|I_n|$-neighborhood N_n of I_n has uniformly bounded degree and distortion.



The Quadratic Case ($d = 2$)

Let f be a quadratic polynomial with a Siegel disc of rotation number ρ.



Theorem (Petersen)

Let F be a cubic Blaschke product with fixed critical points at 0 and ∞, and a double critical point at 1. Then J_F is locally connected. Consequently, if ρ is of bounded type, then J_f is locally connected.

Theorem (Petersen-Zakeri)

J_f is locally connected for almost every ρ.

Puzzles and Fibers

Denote the puzzle partition of depth n by $\mathcal{T}_n = f^{-n}(\mathcal{T}_0)$, and a puzzle piece of depth n by $P_n \subset \mathbb{C} \setminus \mathcal{T}_n$. If $x \in P_0 \supset P_1 \supset \ldots$, then the *fiber* at x is given by $\mathcal{F}_x = \bigcap_{n=0}^{\infty} P_n$. Recall that if $\mathcal{F}_x = \{x\}$, then J_f is locally connected at x.

Puzzle Annuli

Suppose $P_{n_1} \Supset P_{n_2} \Supset \ldots \ni x$. Denote the ith puzzle annuli by $A_i := P_{n_i} \setminus \overline{P_{n_{i+1}}}$.
Grötzsch inequality: $\sum_{i=1}^{\infty} \operatorname{mod}(A_i) = \infty \implies \bigcap_{i=1}^{\infty} P_{n_i} = \mathcal{F}_x = \{x\}$.
Let $A \subset U$ and $B \subset V$ be topological discs, and let $f : (U, A) \to (V, B)$ be a holomorphic branched covering between respective discs. Then

$$\operatorname{mod}(U \setminus \overline{A}) \geq \frac{1}{\deg f} \operatorname{mod}(V \setminus \overline{B}).$$

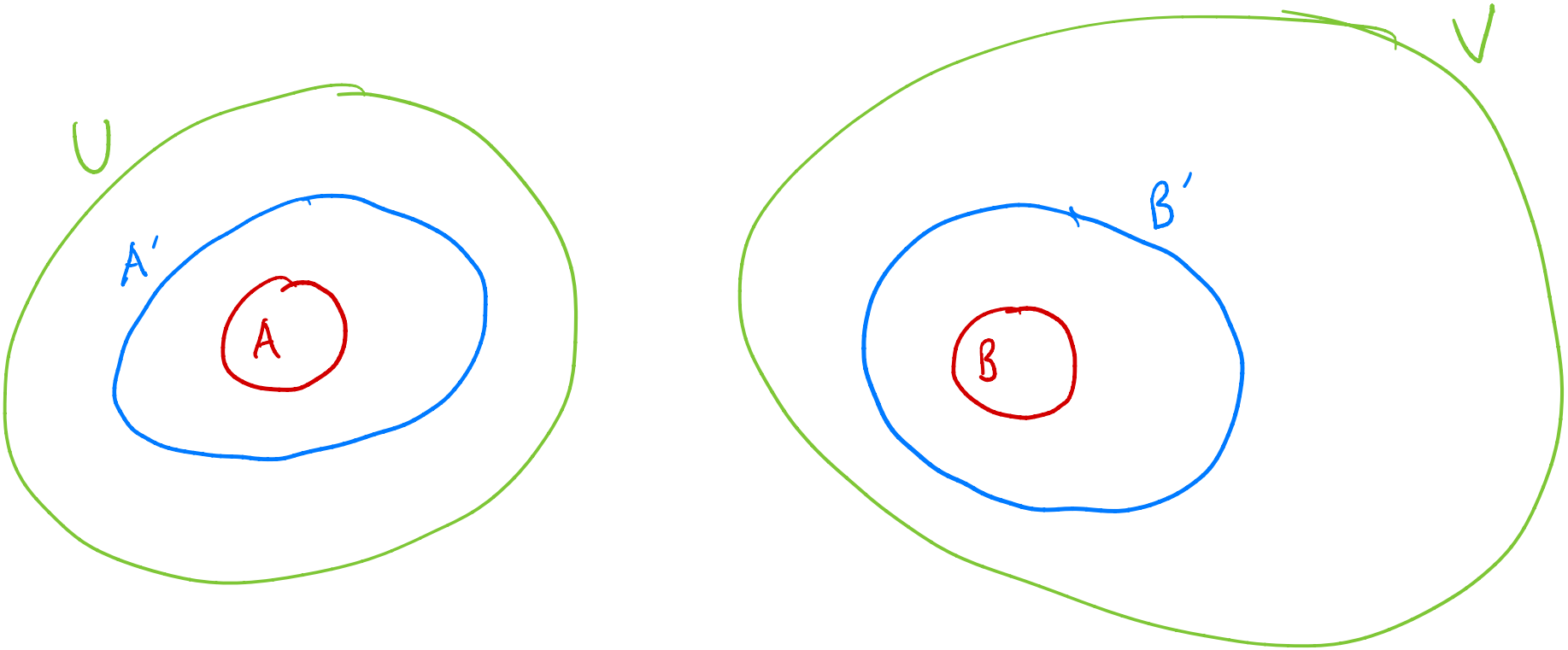
Kahn-Lyubich Covering Lemma

Let $A \Subset A' \Subset U$ and $B \Subset B' \Subset V$ be topological discs, and let $f : (U, A', A) \to (V, B', B)$ be a holomorphic branched covering between respective discs. Denote $D = \deg f \geq d = \deg(f|_{A'})$. Suppose for some $\eta > 0$, we have:

$$\operatorname{mod}(B' \setminus \overline{B}) > \eta \operatorname{mod}(U \setminus \overline{A}).$$

Then there exists $\epsilon = \epsilon(\eta, D) > 0$ such that

$$\operatorname{mod}(U \setminus \overline{A}) > \epsilon \qquad \text{or} \qquad \operatorname{mod}(U \setminus \overline{A}) > \frac{\eta}{2d^2} \operatorname{mod}(V \setminus \overline{B}).$$

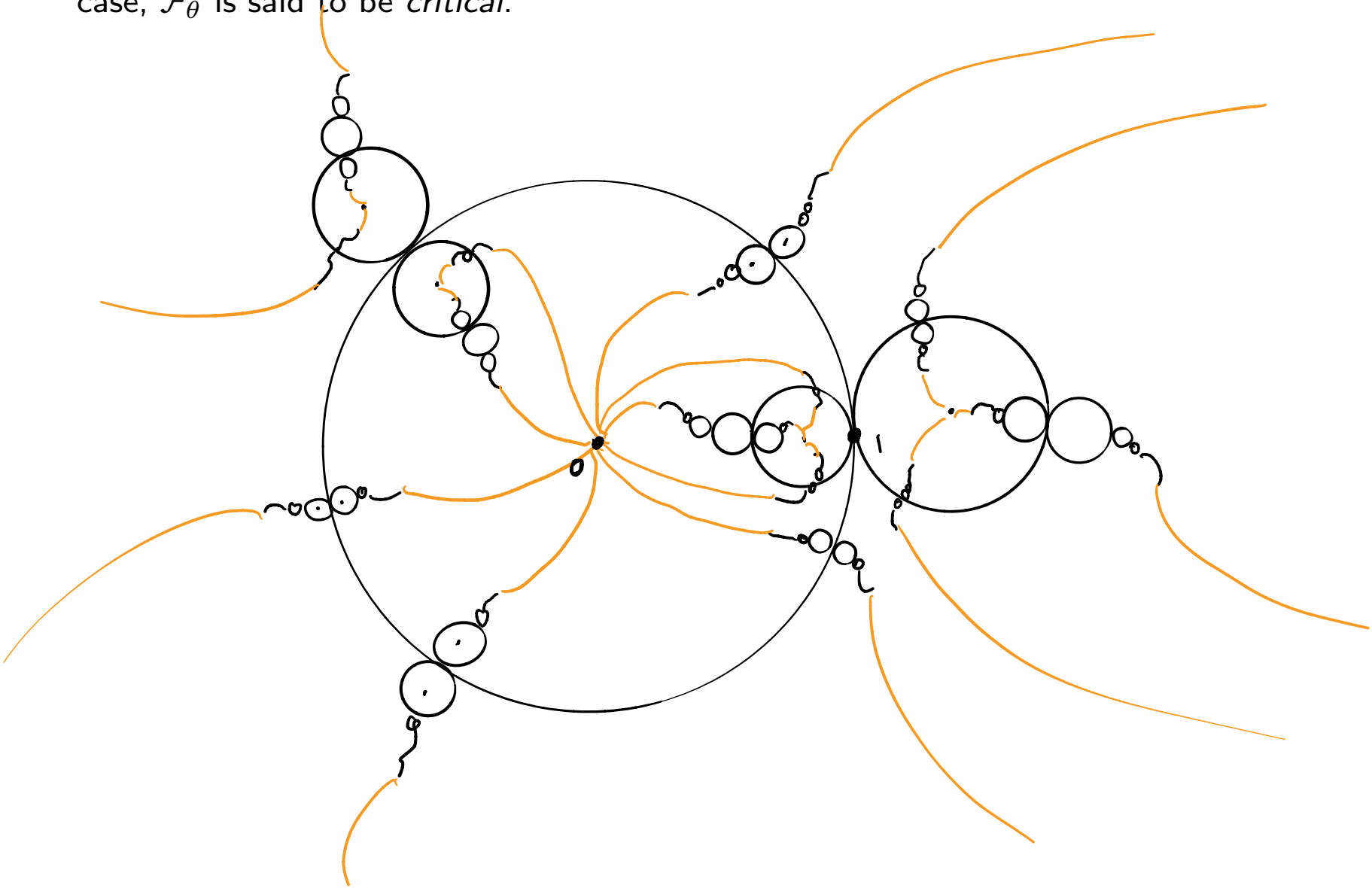


Applying the Covering Lemma

Suppose that $f^{r_i}(P_{n_i}) = P_{n_{i-1}}$, where r_i is the first return time of x to P_{n_i}. Since r_i's grow exponentially, we have $r_i > R = r_{i-1} + r_{i-2} + \ldots + r_{i-K+1}$. Consider

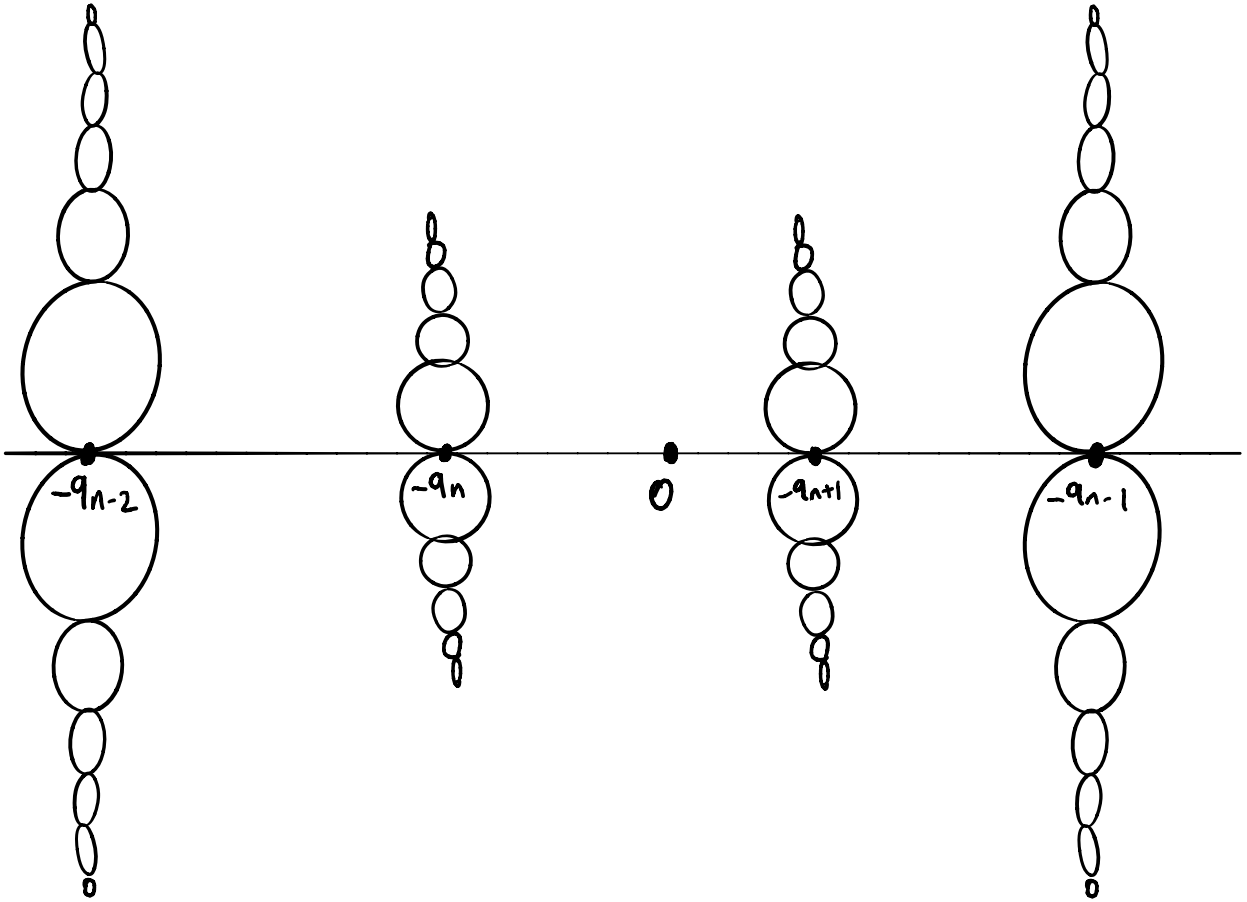
$$(P_{n_{i-1}}, P_{n_i}, P_{n_{i+1}}) \xrightarrow{f^R} (P_{n_{i-K}}, f^R(P_{n_i}), f^R(P_{n_{i+1}})).$$

A fiber can only intersect $\partial\mathbb{D}$ at one point. A fiber $\mathcal{F}_\theta$ for which $\mathcal{F}_\theta \cap \partial\mathbb{D} = \{e^{\theta i}\}$ is said to be *at height* 0. A critical point c is *at height* 0 if $c \in \mathcal{F}_\theta$ for some θ. In this case, $\mathcal{F}_\theta$ is said to be *critical*.



Puzzles with Slits

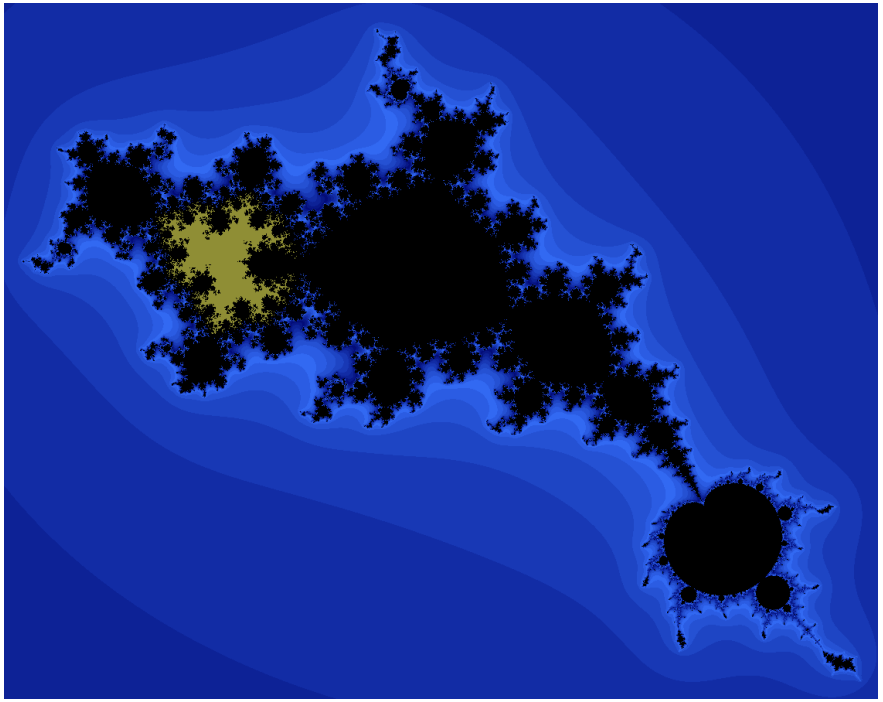
Near Degenerate Modulus and Slits



Cubic Siegel Families

For bounded type ρ, let $\mathcal{C}_\rho$ be the family of cubic polynomials f_a, $a \in \mathbb{C}^*$, that has a Siegel fixed point at 0 of rotation number ρ, and critical points at 1 and a. Let $\mathcal{M}_\rho \subset \mathcal{C}_\rho$ be the connectedness locus. Define the Zakeri curve as

$$\mathcal{Z}_\rho := \{a \in \mathbb{C}^* \mid 1, a \in \partial\Delta_{f_a}\} \subset \mathcal{M}_\rho.$$



Theorem (Yampolsky-Y.)

$\mathcal{M}_\rho$ is locally connected at every point in $\mathcal{Z}_\rho$.