Polynomials with Bounded Type Siegel Disks

Jonguk Yang, Stony Brook University

On Geometric Complexity of Julia Sets II, August 2020
Consider a polynomial $f$ of degree $d \geq 2$ that has a Siegel point at 0 with rotation number $\rho$. Let $0 \in \Delta_f$ be the Siegel disk, $A_f^\infty$ the attracting basin of infinity, $K_f$ the filled Julia set, and $J_f$ the Julia set. WLOG, assume

- 0 is fixed.
- $K_f$ is connected.
Statement of the Theorem

Theorem (Shishikura, Zhang*)

Suppose $\rho$ is of bounded type. Then $\partial \Delta_f$ is a quasicircle that contains at least one critical point. Consequently, $\phi_0$ extends homeomorphically to $\partial \Delta_f$.

* Zhang proved the theorem for all rational maps.

Recall that $\phi^{-1}_\infty$ extends continuously to $\partial \mathbb{D}$ $\iff$ $J_f$ is locally connected.

Theorem (Y.)

Suppose $\rho$ is of bounded type. Then $J_f$ is locally connected at every point in $\partial \Delta_f$.

The analogous theorem for attracting components was proved by Kozlovski-van Strien, and for parabolic components by Roesch-Yin.
Suppose $\rho$ is of bounded type. Then there exists a Blaschke product $F$ of degree $2d - 1$ such that $f$ is quasiconformally conjugate to the modified Blaschke product $\tilde{F}$ obtained from $F$ via the Douady-Ghys surgery.

Henceforth, we assume that $F$ has a critical point at 1.
Consider an analytic circle homeomorphism $F$ with rotation number $\rho$. Choose $x_0 \in \partial \mathbb{D}$. Let $q_n$ be the $n$th closest return time, and let $l_n$ be the $n$th closest return arc bounded by $x_0$ and $x_{q_n} = F^{q_n}(x_0)$. The $n$th dynamic partition of $\partial \mathbb{D}$ is defined as

$$\mathcal{I}_n = \{F^i(l_n) \mid 0 \leq i < q_{n+1}\} \cup \{F^i(l_{n+1}) \mid 0 \leq i < q_n\}.$$
Real A Priori Bounds

Theorem (Herman)

For all adjacent arcs $I, J \in \mathcal{I}_n$, we have $C^{-1}|J| < |I| < C|J|$ for some uniform $C > 1$. Moreover, there exists a uniform constant $\lambda > 0$ such that $F^{q_{n+1}}$ restricted to the $\lambda|I_n|$-neighborhood $N_n$ of $I_n$ has uniformly bounded degree and distortion.
The Quadratic Case ($d = 2$)

Let $f$ be a quadratic polynomial with a Siegel disc of rotation number $\rho$.

**Theorem (Petersen)**

Let $F$ be a cubic Blaschke product with fixed critical points at 0 and $\infty$, and a double critical point at 1. Then $J_F$ is locally connected. Consequently, if $\rho$ is of bounded type, then $J_f$ is locally connected.

**Theorem (Petersen-Zakeri)**

$J_f$ is locally connected for almost every $\rho$. 
Denote the puzzle partition of depth \( n \) by \( \mathcal{T}_n = f^{-n}(\mathcal{T}_0) \), and a puzzle piece of depth \( n \) by \( P_n \subseteq \mathbb{C} \setminus \mathcal{T}_n \). If \( x \in P_0 \supset P_1 \supset \ldots \), then the fiber at \( x \) is given by \( \mathcal{F}_x = \bigcap_{n=0}^{\infty} P_n \). Recall that if \( \mathcal{F}_x = \{x\} \), then \( J_f \) is locally connected at \( x \).
Suppose $P_{n_1} \supset P_{n_2} \supset \ldots \supset \times$. Denote the $i$th puzzle annuli by $A_i := P_{n_i} \setminus \overline{P_{n_{i+1}}}$. Grötzsch inequality: $\sum_{i=1}^{\infty} \mod(A_i) = \infty \implies \bigcap_{i=1}^{\infty} P_{n_i} = F_{\times} = \{x\}$. Let $A \subset U$ and $B \subset V$ be topological discs, and let $f : (U, A) \to (V, B)$ be a holomorphic branched covering between respective discs. Then

$$\mod(U \setminus \overline{A}) \geq \frac{1}{\deg f} \mod(V \setminus \overline{B}).$$
Let $A \subset A' \subset U$ and $B \subset B' \subset V$ be topological discs, and let $f : (U, A', A) \to (V, B', B)$ be a holomorphic branched covering between respective discs. Denote $D = \deg f \geq d = \deg(f|_{A'})$. Suppose for some $\eta > 0$, we have:

$$\text{mod}(B' \setminus B) > \eta \text{mod}(U \setminus A).$$

Then there exists $\epsilon = \epsilon(\eta, D) > 0$ such that

$$\text{mod}(U \setminus A) > \epsilon \quad \text{or} \quad \text{mod}(U \setminus A) > \frac{\eta}{2d^2} \text{mod}(V \setminus B).$$
Suppose that $f^{r_i}(P_{n_i}) = P_{n_{i-1}}$, where $r_i$ is the first return time of $x$ to $P_{n_i}$. Since $r_i$'s grow exponentially, we have $r_i > R = r_{i-1} + r_{i-2} + \ldots + r_{i-K+1}$. Consider

$$(P_{n_{i-1}}, P_{n_i}, P_{n_{i+1}}) \xrightarrow{f^R} (P_{n_{i-K}}, f^R(P_{n_i}), f^R(P_{n_{i+1}})).$$
A fiber can only intersect $\partial \mathbb{D}$ at one point. A fiber $\mathcal{F}_\theta$ for which $\mathcal{F}_\theta \cap \partial \mathbb{D} = \{e^{\theta i}\}$ is said to be at height 0. A critical point $c$ is at height 0 if $c \in \mathcal{F}_\theta$ for some $\theta$. In this case, $\mathcal{F}_\theta$ is said to be critical.
Near Degenerate Modulus and Slits
For bounded type $\rho$, let $\mathcal{C}_\rho$ be the family of cubic polynomials $f_a$, $a \in \mathbb{C}^*$, that has a Siegel fixed point at 0 of rotation number $\rho$, and critical points at 1 and $a$. Let $\mathcal{M}_\rho \subset \mathcal{C}_\rho$ be the connectedness locus. Define the Zakeri curve as

$$\mathcal{Z}_\rho := \{ a \in \mathbb{C}^* \mid 1, a \in \partial \Delta_{f_a} \} \subset \mathcal{M}_\rho.$$