Thermodynamic formalism for coarse expanding dynamical systems

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Geometric complexity of Julia sets, 2020

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Holomorphic dynamics

Motivation coming from dynamics of rational maps of the Riemann sphere-thermodynamic formalism and statistical laws for natural invariant measures. [works by Denker, Przytycki, Rivera- Letelier, Szostakiewicz, Urbański, Zdunik..- the list of authors is not complete]

Recent development- continuous branched coverings of S^2

Expanding Thurston maps- studied recently by M. Bonk, D. Meyer in

[M. Bonk, D. Meyer, *Expanding Thurston maps*, Mathematical Surveys and Monographs 225, American Mathematical Society, 2017.]

Among other things, Bonk and Meyer construct and study the measure of maximal entropy for Thurston expanding maps.

Thermodynamic formalism for Thurston expanding maps- continued

In a recent series of papers, Zhiqiang Li works out the thermodynamic formalism for expanding Thurston maps, with respect to the *visual metric*.

Z. Li, *Weak expansion properties and large deviation principles for expanding Thurston Maps*, Adv. Math. 285 (2015), 515–567.

Z. Li, *Periodic points and the measure of maximal entropy of an expanding Thurston map*, Trans. Amer. Math. Soc. 368 (2016), 8955–8999.

Z. Li, *Equilibrium states for expanding Thurston maps* , Comm. Math. Phys. 357 (2018), 811–872.

Z. Li, *Ergodic theory of expanding Thurston maps*, The Atlantis Series in Dynamical Systems, volume 4, Atlantis Press (Springer), 2017.

Coarse expanding maps- works by P.Haïssisky and K. Pilgrim

P. Haïssinsky and K.Pilgrim define more generally the concept of *coarse expanding conformal* (cxc) system, developing in an axiomatic way the notion of expansion for maps of general metric spaces. In particular, they prove for cxc systems existence and uniqueness of the measure of maximal entropy.

P. Haïssinsky, K. M. Pilgrim, *Coarse expanding conformal dynamics*, Astérisque 325 (2009), viii+139 pp.

P. Haïssinsky, K. M. Pilgrim, *Examples of coarse expanding conformal maps*, Discrete Contin. Dyn. Syst. 32, no. 7 (2012), 2403-2416.

The goal of our research

The goal of our research is to develop the thermodynamic formalism (in particular, prove existence and uniqueness of equilibrium states and statistical laws) for a general class of dynamical systems, which we call *weakly coarse expanding*. These systems are continuous finite branched coverings of locally connected topological spaces which are expanding in a weak metric sense, and generalize most cases discussed by Bonk –Meyer, Li and Haïssinsky-Pilgrim.

In particular, they need not be postcritically finite, and the metric we consider need not be the visual metric, but it could be more generally an exponentially contracting metric. **Neither conformality, nor holomorphy or smoothness is assumed. In particular, periodic branch (critical) points may be repelling despite the local degree at them being bigger than 1.**

Degree of a map

Let $f : Y \to Z$ be a continuous map between locally compact Hausdorff topological spaces. The *degree* of f is defined as $\deg(f) := \sup\{\# f^{-1}(z) \ : \ z \in Z\}$. Given a point $y \in Y$, the *local degree* of f at y is

$$\deg(f;y) := \inf_U \sup \#\{f^{-1}(z) \cap U \ : \ z \in f(U)\}$$

where U ranges over all open neighborhoods of y. A point y is *critical* if $\deg(f;y) > 1$.

Definition of a finite branched cover

The map $f : Y \to Z$ is a *finite branched cover* of degree d if $\deg(f) = d < \infty$ and the two following conditions hold:

1. for any $z \in Z$,
$$\sum_{y \in f^{-1}(z)} \deg(f; y) = \deg(f)$$
2. for $y_0 \in Y$ there are compact nenghbourhoods U, V of y_0 and $f(y_0)$ such that
$$\forall z \in V \sum_{y \in U, f(y) = z} \deg(f; y) = \deg(f; y_0).$$

branch set : $B_f := \{y \in Y \ : \ \deg(f; y) > 1\}$, *branch values*: $V_f = f(B_f)$.

Properties of finite branched covers: A finite branched cover is open, closed, onto and proper. Furthermore, B_f and V_f are closed and nowhere dense. A finite branched cover f is a local homeomorphism away from critical points; i.e., for any $y \notin B_f$ there exists an open set U which contains y and such that $f : U \to f(U)$ is a homeomorphism.

Branched system, repellor, postsingular set

Let W_1, W_0 be two locally compact, locally connected, Hausdorff topological spaces, each with finitely many connected components, and suppose that W_1 is an open subset of W_0 and the closure of W_1 is compact.
We define a *system* as a triple (f, W_0, W_1), where W_0, W_1 satisfy the hypotheses above, and $f : W_1 \to W_0$ is a finite branched cover.

Definition

We define the *repellor* X of the system (f, W_0, W_1) as

$$X := \bigcap_{n=0}^{\infty} f^{-n}(W_1).$$

By definition, $f(X) \subseteq X$ and $X = f^{-1}(X) \cap W_1$. Moreover, we define the *post-branch set* as

$$P_f := X \cap \bigcup_{n>0} V_{f^n}.$$

Note that we do *not* take the closure of the post-branch set.

[Expansion] and [LEO]

The following topological definition of expansion was proposed by Haïssinsky and Pilgrim. Let $\mathcal{U}_0$ be a finite cover of X by connected, open subsets of W_1 whose intersection with X is not empty. For each n, we define $\mathcal{U}_n$ as the open cover whose elements are the connected components of $f^{-n}(U)$ where U belongs to $\mathcal{U}_0$.

Definition

A system (f, W_0, W_1) satisfies the [Expansion] axiom if there exists a finite cover $\mathcal{U}_0$ of X such that the following holds: for any open cover $\mathcal{Y}$ of X by open subsets of W_0, there exists N such that for all $n \geq N$ each element of $\mathcal{U}_n$ is contained in some element of $\mathcal{Y}$.

If W_0 is equipped with a metric, this axiom implies that the diameter of the cover $\mathcal{U}_n$ tends uniformly to zero as $n \to \infty$.

Definition

A system (f, W_0, W_1) is *locally eventually onto* if for any $x \in X$ and any neighborhood W of x in X there is some n with $f^n(W) = X$.

Wekly coarse expanding systems

Definition

A system (f, W_0, W_1) is *weakly coarse expanding* if:

1. it satisfies the [Expansion] axiom;
2. it is locally eventually onto;
3. the branch set B_f is finite;
4. the repellor X is not a single point.

Note that if $W_0 \subseteq S^2$ is an open subset of the 2-sphere, then the set B_f is always finite.

Note that we do not assume [Degree] property required by Haïssinsy and Pilgrim, i.e., we allow local degree of iterates of f to be unbounded.

Exponential contracting metric

A very important property of coarse expanding systems is that we can find a metric so that preimages shrink exponentially fast.

Theorem

Suppose $f : W_1 \to W_0$ *is a finite branched cover and axiom* [Expansion] *holds. Then there exists a metric* ρ *on* X *and constants* $C > 0$, $\theta < 1$ *such that for all* $n \geq 0$

$$\sup_{U \in \mathcal{U}_n} \operatorname{diam}_\rho(U) \leq C\theta^n.$$

We call a metric ρ which satisfies the above property an *exponentially contracting* metric. An important example (with additional properties) is the *visual metric* constructed by Haïssinsy and Pilgrim.

We give a different proof of existence of this measure using *Frink's Metrization Lemma*.

Frink's Lemma

Lemma (Theorem of A.H. Frink)

Let X be a topological space, and let $(\Omega_n)_{n\geq 0}$ be a sequence of open neighborhoods of the diagonal $\Delta \subseteq X \times X$, such that

(a)
$$\Omega_0 = X \times X$$

(b)
$$\bigcap_{n=0}^{\infty} \Omega_n = \Delta$$

where Δ is the diagonal in $X \times X$.

(c) *For any $n \geq 1$,*
$$\Omega_n \circ \Omega_n \circ \Omega_n \subseteq \Omega_{n-1}$$
where $\circ$ is the composition in the sense of relations: i.e.,
$R \circ S = \{(x, y) \in X \times X \ : \ \exists z \in X \text{ s.t. } (x, z) \in R \text{ and } (z, y) \in S\}$.

Then there exists a metric ρ on X, compatible with the topology, such that

$$\Omega_n \subseteq \{(x, y) \in X \times X \ : \ \rho(x, y) < 2^{-n}\} \subseteq \Omega_{n-1}$$

for any $n \geq 1$.

here, we recall definition of an equilibrium state

Let $f : X \to X$ be a continuous map of a compact metric space, μ– a probability invariant measure for f. A
Consider a continuous function $\varphi : X \to \mathbb{R}$, which we call a *potential*. The *topological pressure* of f with potential φ is defined as

$$P_{top}(\varphi) = \sup_{\mu \in M(f)} \left\{ h_\mu(f) + \int_X \varphi \, d\mu \right\}.$$

An f-invariant probability measure μ on X is an *equilibrium state* for φ if it realizes the supremum, namely if

$$P_{top}(\varphi) = h_\mu(f) + \int_X \varphi \, d\mu.$$

We shall consider a weakly expanding system $f : W_1 \to W_0$, and study the equilibrium states on its repellor X.

Coding tree

The key point in our approach is that one constructs a semiconjugacy of a weakly coarse expanding system of degree d to the shift map on d symbols.
Let $\Sigma := \{1, \ldots, d\}^{\mathbb{N}}$ be the space of infinite sequences of d symbols, and $\sigma : \Sigma \to \Sigma$ the left shift.

Proposition.

Let $f : W_1 \to W_0$ be a weakly coarse expanding system of degree d and assume W_0 is strongly path connected. Then there exists a Hölder continuous semiconjugacy $\pi : \Sigma \to X$ such that $\pi \circ \sigma = f \circ \pi$. Moreover, if f is locally eventually onto, then π is surjective.

(A space $\mathcal{X}$ is strongly path connected if any countable subset $S \subset \mathcal{X}$, the space $\mathcal{X} \setminus S$ is path connected.)

Construction of a tree

The construction is based on the idea of “geometric coding tree" as in earier works of [F.P], [F.P, A.Z], [PUZ]... Namely, pick $w \in X \setminus P_f$, and let $w_1, \dots, w_d$ be its preimages. For each $i = 1, \dots, d$, choose a continuous path γ_i in W_0 connecting w and w_i and avoiding P_f. This exists since the space is strongly path connected and the set P_f is countable. We write $f_i : [0, 1] \to W_0$ to denote the continuous map whose image is $f_i([0, 1]) = \gamma_i \subset W_0$.
For each sequence $\alpha = (i_1, i_2, \dots) \in \Sigma$, define $z_n(\alpha)$ by letting $z_0(\alpha) = w_{i_1}$ and $z_n(\alpha)$ for $n \geq 1$ inductively as follows. Let $\gamma(z_n)$ be a curve which is the branch of $f^{-(n-1)}(\gamma_{i_n})$ such that one of its ends is $z_{n-1}(\alpha)$. Then define $z_n(\alpha)$ as the other end of $\gamma(z_n)$.

Now, using the exponential contraction property one can check that

$$\rho(z_n(\alpha), z_{n-1}(\alpha)) \preceq \theta^n,$$

and hence

$$\lim_{n\to\infty} z_n(\alpha)$$

exists, and we define $\pi(\alpha)$ as the limit. By construction the map π satisfies $f \circ \pi = \pi \circ \sigma$.

The coding is *almost* injective.

For every $x \in X$ denote by $S_{n,x}$ cardinality of the collection of cylinders $C_n \subset \Sigma$ of length n such that $\pi^{-1}(x) \cap C_n \neq \emptyset$.

Theorem

Suppose that no critical point is periodic. Then

$$\lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} S_{n,x} = 0.$$

Application: No entropy drop

Proposition

Let μ be a σ-invariant measure on the symbolic space Σ, and let $\nu = \pi_\star \mu$ be the pushforward measure on X. Then

$$h_\mu(\sigma) = h_\nu(f).$$

Proof. Let $\mathcal{A}^n$ denote the partition of Σ into cylinders of depth n, let θ be the partition in preimages of points under π, and for ν-a.e. $x \in X$ let μ_x be the conditional measure on the fiber over x. By definition of relative entropy

$$\begin{aligned} h_\mu(\sigma|f) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{A}^n|\theta) \\ &= \lim_{n\to\infty} \frac{1}{n} \int_X d\nu(x) \sum_{a\in\mathcal{A}^n} -\mu_x(a) \log \mu_x(a) \end{aligned}$$

and, by the comparison between measure-theoretic and conditional entropy,

$$\leq \limsup_{n\to\infty} \frac{1}{n} \int_X \log S_{n,x} \, d\nu(x)$$

hence the claim follows from the previous estimate of cardinality od $S_{n,x}$.

Theorem

Let $f : W_1 \to W_0$ be a weakly coarse expanding dynamical system without periodic critical points, let X be its repellor, ρ an exponentially contracting metric on X compatible with the topology, and let $\varphi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous function. Then:
(1) there exists a unique equilibrium state μ_φ for φ on X.
(2)Let $\psi : (X, \rho) \to \mathbb{R}$ be a Hölder continuous observable, and denote

$$S_n\psi(x) := \sum_{k=0}^{n-1} \psi(f^k(x)).$$

Then there exists the finite limit

$$\sigma^2 := \lim_{n\to\infty} \frac{1}{n} \int_X \left(S_n\psi(x) - n \int \psi \, d\mu_\varphi \right)^2 d\mu_\varphi \geq 0$$

such that the following statistical laws hold....

Central Limit Theorem:

If $\sigma > 0$, we have for any $a < b$

$$\mu_\varphi\left(\left\{x \in X \ : \ \frac{S_n\psi(x) - n\int_X \psi\, d\mu_\varphi}{\sqrt{n}} \in [a,b]\right\}\right) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\int_a^b e^{-t^2/2\sigma^2}\, dt$$

as $n \to \infty$. If $\sigma = 0$, one has convergence in probability to the Dirac δ-mass at 0.

Law of Iterated Logarithm: For μ_φ-a.e. $x \in X$,

$$\limsup_{n\to\infty} \frac{S_n\psi(x) - n\int_X \psi\, d\mu_\varphi}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2}.$$

Exponential Decay of Correlations: For any μ_φ–integrable function $\chi : X \to \mathbb{R}$ there exist constants $\alpha > 0$ and $C \geq 0$ such that for any $n \geq 0$,

$$\left|\int_X \psi \cdot (\chi \circ f^n)\, d\mu_\varphi - \int_X \psi\, d\mu_\varphi \cdot \int_X \chi\, d\mu_\varphi\right| \leq Ce^{-n\alpha}.$$

Large Deviations: For every $t \in \mathbb{R}$, we have that

$$\lim_{n\to\infty} \frac{1}{n}\log \mu_\varphi\left(\left\{x \in X : \operatorname{sgn}(t)S_n\psi(x) \geq \operatorname{sgn}(t)n\int_X \psi\, d\mu_{\varphi+t\psi}\right\}\right)$$

$$= -t\int_X \psi\, d\mu_{\varphi+t\psi} + P_{top}(\varphi + t\psi) - P_{top}(\varphi).$$

Moreover, $\sigma = 0$ if and only if there exists a continuous $u : X \to \mathbb{R}$ such that

$$\psi - \int_X \psi \, d\mu_\varphi = u \circ f - u.$$

Finally, $\mu_{\varphi_1} = \mu_{\varphi_2}$ if and only if there exists $K \in \mathbb{R}$ and a continuous $u : X \to \mathbb{R}$ such that

$$\varphi_1 - \varphi_2 = u \circ f - u + K.$$

The function u is Hölder continuous with respect to the visual metric.

Now, we allow critical periodic points in the repeller X

Theorem

If $f : W_1 \to W_0$ is a weakly coarse expanding system and $W_0 \subseteq S^2$ is an open subset of the 2-sphere, then all claims of the previous Theorem hold even if there are periodic critical points.

The main step in the proof of the above theorem is the following

Theorem

Let $f : W_1 \to W_0 \subseteq S^2$ be a weakly coarse expanding map on an open subset of the 2-sphere, and let ρ be an exponentially contracting metric on X. Then there exists a strongly path connected space $\widetilde{W}_0$ and a weakly coarse expanding system $g : \widetilde{W}_1 \to \widetilde{W}_0$ without periodic critical points with repellor Y and a metric ρ' on Y which is exponentially contracting with respect to g, and there is a continuous map $\pi : \widetilde{W}_0 \to W_0$ such that $\pi \circ g = f \circ \pi$.

Construction of $\tilde{W}_0$

Initial observation:

Lemma

Let $f : W_1 \to W_0 \subseteq S^2$ be a weakly coarse expanding map, and let $p \in X$ be a fixed critical point. Then there exists $d \in \mathbb{Z} \setminus \{0\}$, a neighborhood U of p, $\lambda > 1$ and a homeomorphism $h : \overline{\mathbb{D}} \to \overline{U}$ such that $f \circ h = h \circ g$ where $g : \overline{\mathbb{D}}_{\lambda^{-1}} \to \overline{\mathbb{D}}$ is defined as

$$g(re^{i\theta}) = \lambda re^{id\theta}$$

for any $r \leq 1$, $\theta \in \mathbb{R}$.

Construction of the blowup.

Let $\mathcal{C}$ denote the (finite) set of periodic critical points which lie in X, and let $\mathcal{E} = \bigcup_{n \geq 0} f^{-n}(\mathcal{C})$. We define a space $\widetilde{S}$ which is given by blowing up every point of $\mathcal{E}$ to a circle. For each $q \in \mathcal{E}$ let S_q be a copy of S^1. The space $\widetilde{S}$ is defined as a *set* as

$$\widetilde{S} = (S^2 \setminus \mathcal{E}) \sqcup \bigsqcup_{q \in \mathcal{E}} S_q.$$

There is a natural way of extending f to $g : \tilde{W}_1 \to \tilde{W}_0$, and one can define a topology on $\tilde{W}_0$ which makes this map continuous, and, moreover, the natural projection $\Pi : \tilde{W}_0 \to W_0$ is continuous.
Next, we prove that this new system

$$(\tilde{W}_1, \tilde{W}_0, g)$$

is a weakly coarse expanding system without periodic critical points.

Proposition

The expanding system $g : \widetilde{W}_1 \to \widetilde{W}_0$ can be continuously embedded into the sphere S^2, where Y becomes a repellor for the extended system. More precisely, there exists a continuous embedding $\iota : \widetilde{W}_0 \to S^2$, a continuous map $g' : S^2 \to S^2$ with $g' \circ \iota = \iota \circ g$ and an open set W_0' which contains $\iota(\widetilde{W}_0)$ so that $\iota(Y) = \bigcap_{n \geq 0} (g')^{-n}(W_0')$.