Working seminar on arXiv:2002.09015 [math.KT]

(*a.k.a.* what Piotr and I did in Naples before COVID-19)

Francesco D'Andrea

06/04/2020

Skype meetings on C*-algebras 2020

Summary

Plan of the talk: go through some of the proofs in arXiv:2002.09015 [math.KT].

1. Three preliminary lemmas
2. Milnor approach
3. Comparison theorem: from q- to multipullback

Three preliminary lemmas

Lemma 1: attaching pullbacks.

Given a commutative diagram of C*-algebras and morphisms:

$$\begin{array}{ccccc} A & \longrightarrow & B & \xrightarrow{f} & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \xrightarrow{g} & F \end{array}$$

if the two inner squares are pullbacks, then so is the outer rectangle;

if the right square and the outer rectangles are pullbacks, so is the left square.

Lemma 2: gauging $\mathrm{U}(1)$-actions.

In the above diagram, suppose f and g are isomorphisms.

Then the left square is a pullback $\iff$ the outer rectangle is a pullback.

Three preliminary lemmas

Remark (Waldhausen category)

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Three preliminary lemmas

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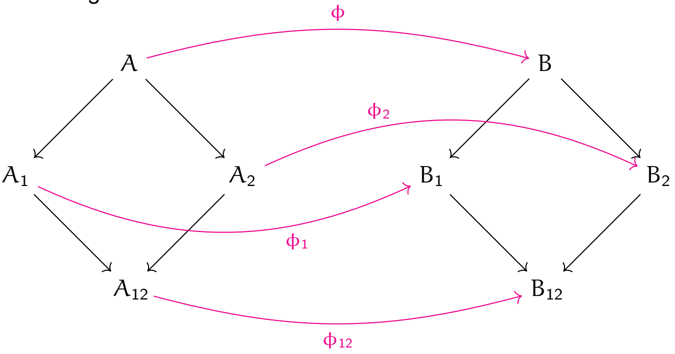
Lemma 3: comparison. [Farsi, Hajac, Maszczyk, Zieliński, 2017]

In a commutative diagram



if the squares are pullback diagrams and ϕ, ϕ_1, ϕ_2 are weak equivalences, then so is ϕ.

Multipullback quantum spaces

- $C(S_H^{2n+1})$ is generated by commuting partial isometries $s_0, \dots, s_n$ with relation

$$\prod_{i=0}^{n}(1 - s_i s_i^*) = 0$$

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- $t :=$ right unilateral shift on $\ell^2(\mathbb{N})$ and

$$t_i := \underbrace{1 \otimes \dots \otimes 1}_{i \text{ times}} \otimes t \otimes \underbrace{1 \otimes \dots \otimes 1}_{n-i \text{ times}} \qquad \forall\, i = 0, \dots, n.$$

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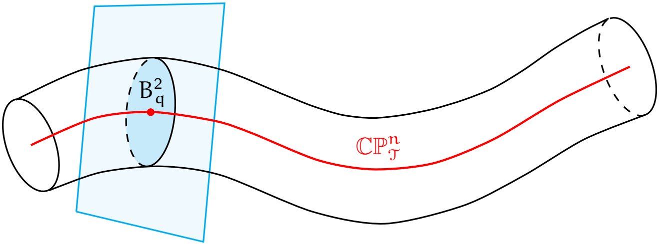
- $C(\mathbb{CP}_{\mathcal{T}}^n) := C(S_H^{2n+1})^{U(1)}$.

- There is a $U(1)$-equivariant short exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N}^{\otimes n+1})) \longrightarrow \mathcal{T}^{\otimes n+1} \xrightarrow{\sigma_n} C(S_H^{2n+1}) \longrightarrow 0$$

where $\boxed{\sigma_n(t_i) := s_i}$ for all $i = 0, \dots, n$.

A tubular neighbourhood theorem



Want to prove ($\forall\ n \geqslant 0$):

Theorem

The map

$$C(\mathbb{CP}^n_{\mathcal{T}}) \xrightarrow{\mathrm{id}\otimes 1_{\mathcal{T}}} C\big(TN(\mathbb{CP}^n_{\mathcal{T}})\big) := \big(C(S_H^{2n+1}) \otimes \mathcal{T}\big)^{U(1)}$$

is a weak equivalence.

Lemma

[Hajac, Nest, Pask, Sims, Zieliński, 2018]

For every $n \geqslant 1$ the diagram

$$
\begin{array}{ccccc}
 & & C(S_H^{2n+1}) & & \\
 & \swarrow^{p_1} & & \searrow^{p_2} & \\
C(S_H^{2n-1}) \otimes \mathcal{T} & & & & \mathcal{T}^{\otimes n} \otimes C(S^1) \\
 & \searrow_{\pi_1} & & \swarrow_{\pi_2} & \\
 & & C(S_H^{2n-1}) \otimes C(S^1) & &
\end{array}
$$

$$p_1(s_i) := \begin{cases} s_i \otimes 1 & \forall\, 0 \leqslant i < n \\ 1 \otimes t & \text{if } i = n \end{cases}$$

$$p_2(s_i) := \begin{cases} t_i \otimes 1 & \forall\, 0 \leqslant i < n \\ 1 \otimes u & \text{if } i = n \end{cases}$$

$$\pi_1 := \mathrm{id} \otimes \sigma_0$$

$$\pi_2 := \sigma_{n-1} \otimes \mathrm{id}$$

is a $U(1)$-equivariant pullback diagram w.r.t. the diagonal $U(1)$-action on each vertex.

We can tensor everywhere by $\mathcal{T}^{\otimes k} =: \mathcal{T}^k$ and get a new pullback diagram:

$$
\begin{array}{ccccc}
 & & C(S_H^{2n+1})_\bullet \otimes \mathcal{T}^k_\bullet & & \\
 & \swarrow^{\,p_1 \otimes \mathrm{id}_{\mathcal{T}^k}} & & \searrow^{\,p_2 \otimes \mathrm{id}_{\mathcal{T}^k}} & \\
\left(C(S_H^{2n-1})_\bullet \otimes \mathcal{T}_\bullet\right) \otimes \mathcal{T}^k_\bullet & & & & \mathcal{T}_\bullet^{\otimes n} \otimes C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet \\
 & \searrow_{\,\pi_1 \otimes \mathrm{id}_{\mathcal{T}^k}} & & \swarrow_{\,\pi_2 \otimes \mathrm{id}_{\mathcal{T}^k}} & \\
 & & C(S_H^{2n-1})_\bullet \otimes C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet & &
\end{array}
$$

. . . then gauge the $U(1)$-action using the map:

$$\phi : A_\bullet \otimes C(S^1)_\bullet \otimes B_\bullet \to A \otimes C(S^1)_\bullet \otimes B$$
$$a \otimes f \otimes b \mapsto a_{(0)} \otimes a_{(1)} f b_{(-1)} \otimes b_{(0)}$$

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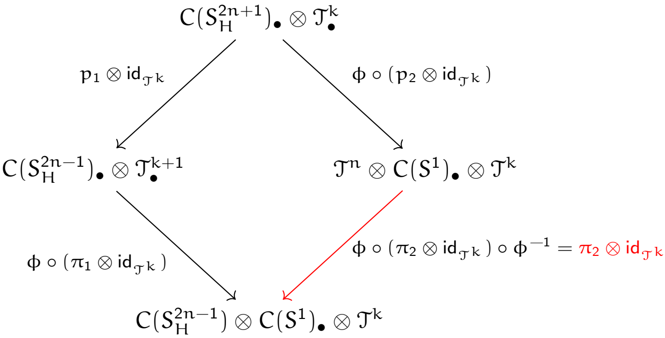
and get the $U(1)$-pullback diagram:



For $k = 0$ and for all $n \geqslant 1$, the $U(1)$-invariant part gives the pullback diagram:

$$
\begin{array}{ccccc}
 & & C(\mathbb{CP}^n_{\mathcal{T}}) & & \\
 & \swarrow^{p_1} & & \searrow^{\phi \circ p_2} & \\
\left(C(S_H^{2n-1}) \otimes \mathcal{T}\right)^{U(1)} & & & & \mathcal{T}^{\otimes n} \\
 & \searrow_{\phi \circ (\mathrm{id} \otimes \sigma_0)} & & \swarrow_{\sigma_{n-1}} & \\
 & & C(S_H^{2n-1}) & &
\end{array}
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$$\begin{array}{ccccc} & & C(\mathbb{CP}^n_{\mathcal{T}}) & & \\ & \swarrow^{p_1} & & \searrow^{\phi \circ p_2} & \\ \big(C(S^{2n-1}_H) \otimes \mathcal{T}\big)^{U(1)} & & & & \mathcal{T}^{\otimes n} \\ & \searrow_{\phi \circ (\mathrm{id} \otimes \sigma_0)} & & \swarrow_{\sigma_{n-1}} & \\ & & C(S^{2n-1}_H) & & \end{array}$$

Recall that: $\big(C(S^{2n-1}_H) \otimes \mathcal{T}\big)^{U(1)} = C\big(TN(\mathbb{CP}^{n-1}_{\mathcal{T}})\big)$

We can now prove the tubular neighbourhood theorem.

Observation

A, B unital with $K_0 = \mathbb{Z}[1]$ and $K_1 = 0 \implies$ any unital $*$-hom $A \to B$ is a weak equivalence.

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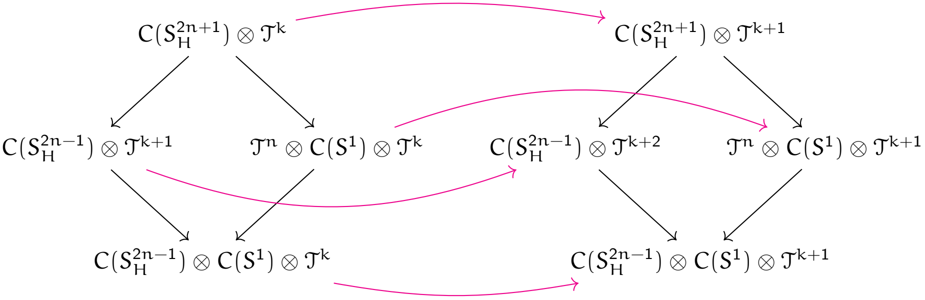
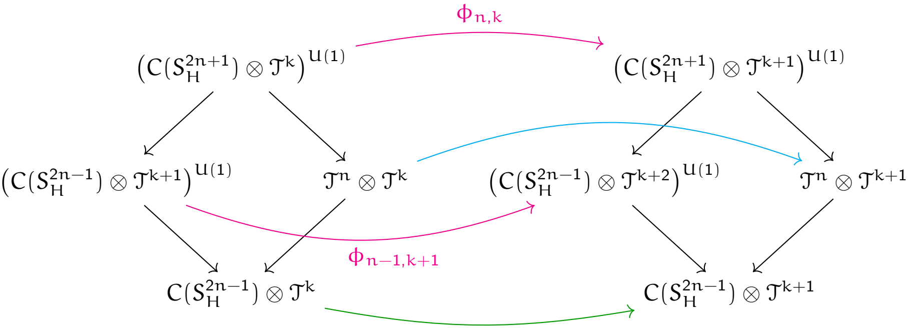
Proof. Consider the commutative diagram:



The squares are pullbacks, the $U(1)$-action is diagonal on top-left vertices and central on bottom-right, horizontal arrows are $\mathrm{id} \otimes 1_{\mathcal{T}}$.

The $U(1)$-invariant part gives:



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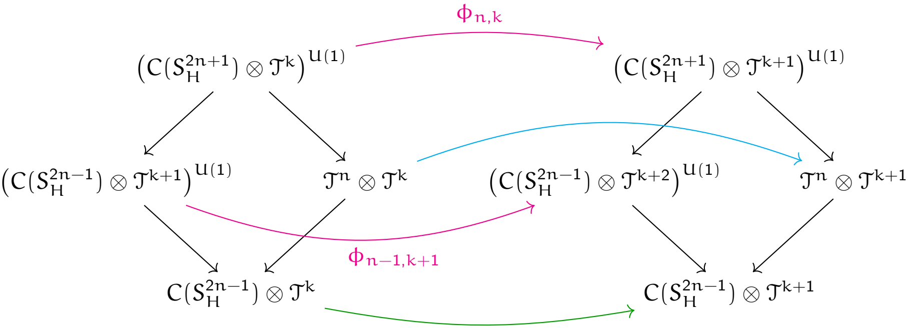

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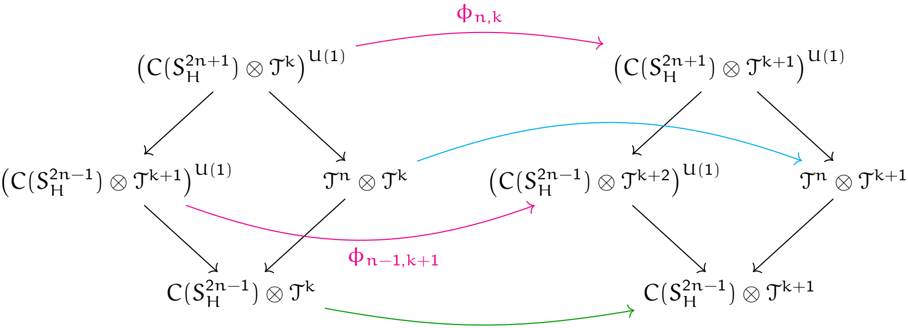

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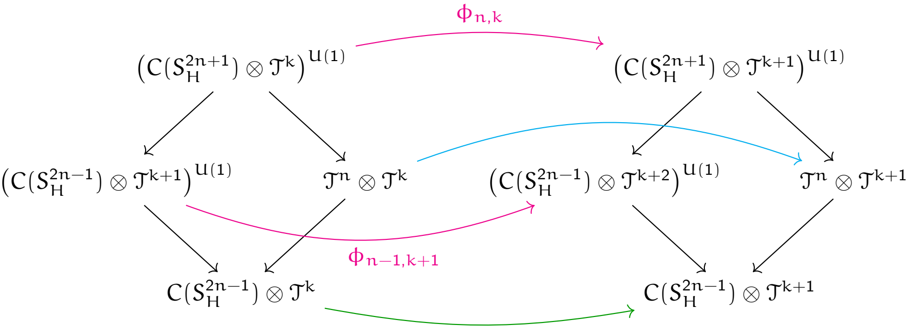

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- If we prove that $\phi_{0,k}$ is a weak equivalence for all $k \geqslant 0$, Lemma 3 $+$ induction on n imply that $\phi_{n,k}$ is a weak equivalence for all $n, k \geqslant 0$.

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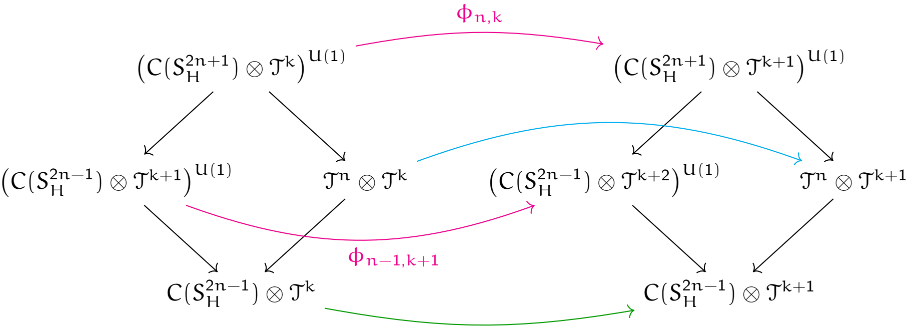

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- Green line $=$ tensor product of weak equivalences ($\mathrm{id}_{C(S^{2n-1})_H}$ and $\mathrm{id}_{\mathcal{T}^k} \otimes 1_{\mathcal{T}}$).
- If we prove that $\phi_{0,k}$ is a weak equivalence for all $k \geqslant 0$, Lemma 3 $+$ induction on n imply that $\phi_{n,k}$ is a weak equivalence for all $n, k \geqslant 0$.
- In particular $\phi_{n,0} = \mathrm{id} \otimes 1_{\mathcal{T}} : C(\mathbb{CP}^n_{\mathcal{T}}) \to C\big(TN(\mathbb{CP}^n_{\mathcal{T}})\big)$ is a weak equivalence.

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$U(1)$-equiv. commutative diagram:

$$\begin{array}{ccc} C(S^1)_\bullet \otimes \mathcal{T}^k_\bullet & \xrightarrow{\mathrm{id}\otimes 1_{\mathcal{T}}} & C(S^1)_\bullet \otimes \mathcal{T}^{k+1}_\bullet \\ \downarrow & & \downarrow \\ C(S^1)_\bullet \otimes \mathcal{T}^k & \xrightarrow{\mathrm{id}\otimes 1_{\mathcal{T}}} & C(S^1)_\bullet \otimes \mathcal{T}^{k+1} \end{array}$$

(red arrows: $a \otimes b \mapsto ab_{(-1)} \otimes b_{(0)}$)

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\downarrow & & \downarrow \\
C(S^1)_\bullet \otimes \mathcal{T}^k & \xrightarrow{\ \mathrm{id} \otimes 1_{\mathcal{T}}\ } & C(S^1)_\bullet \otimes \mathcal{T}^{k+1}
\end{array}$$

$U(1)$-invariant part:

$$\begin{array}{ccc}
\left(C(S^1) \otimes \mathcal{T}^k\right)^{U(1)} & \xrightarrow{\ \phi_{0,k}\ } & \left(C(S^1) \otimes \mathcal{T}^{k+1}\right)^{U(1)} \\
\downarrow & & \downarrow \\
\mathcal{T}^k & \xrightarrow{\ \mathrm{id} \otimes 1_{\mathcal{T}}\ } & \mathcal{T}^{k+1}
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(red arrows: $a \otimes b \mapsto ab_{(-1)} \otimes b_{(0)}$)

Since the vertical arrows are isomorphisms and (by the above Observation) the bottom arrows are weak equivalences, $\phi_{0,k}$ is a weak equivalence. ■

Milnor and K-groups

Using the six-term exact sequence associated to the one-surjective pullback diagram

$$\begin{array}{ccc} & C(\mathbb{CP}^n_{\mathcal{T}}) & \\ \swarrow & & \searrow \\ C\big(TN(\mathbb{CP}^{n-1}_{\mathcal{T}})\big) & & \mathcal{T}^{\otimes n} \\ \searrow & & \swarrow \\ & C(S_H^{2n-1}) & \end{array}$$

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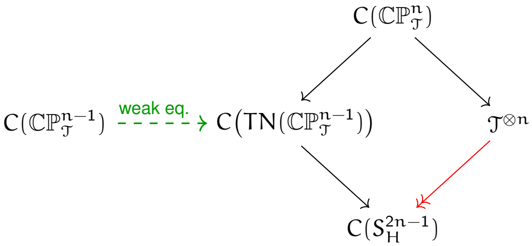

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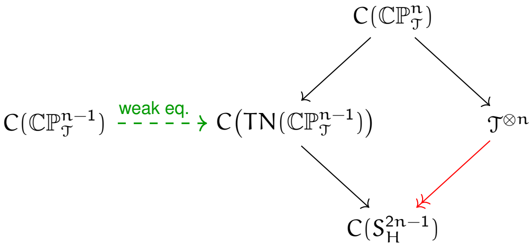

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Proposition

For all $n \geqslant 1$:

$$K_0(C(\mathbb{CP}^n_{\mathcal{T}})) \simeq K_0(C(\mathbb{CP}^{n-1}_{\mathcal{T}})) \oplus \partial\big(K_1(C(S^{2n-1}_H))\big)$$

where ∂ is Milnor connecting homomorphism.

A comparison theorem

For all $n \geqslant 0$ there is a $U(1)$-equivariant weak equivalence:

$$C(S_q^{2n+1}) \to C(S_H^{2n+1})$$

Its restriction and corestriction to fixed point algebras:

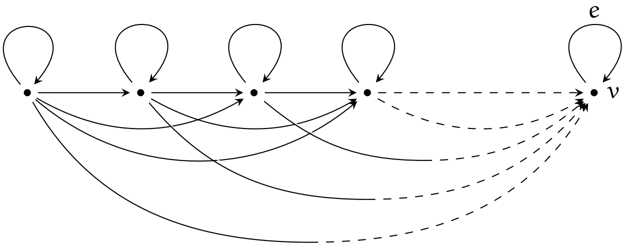
$$C(\mathbb{CP}_q^n) \to C(\mathbb{CP}_{\mathcal{T}}^n)$$

is a weak equivalence as well.

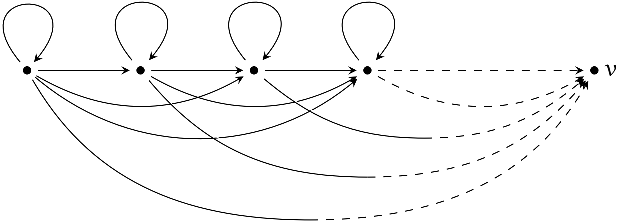
It maps vector bundles associated to $S_q^{2n+1} \xrightarrow{U(1)} \mathbb{CP}_q^n$, generators of $K_0(C(\mathbb{CP}_q^n))$, to v. bundles associated to $S_H^{2n+1} \xrightarrow{U(1)} \mathbb{CP}_{\mathcal{T}}^n$,* proving that the latter generate $K_0(C(\mathbb{CP}_{\mathcal{T}}^n))$.

*Pushforward commutes with association, cf. [Hajac, Maszczyk, 2018].

Morphisms



S_q^{2n+1}



B_q^{2n}

$$C(S_q^{2n+1}) \xrightarrow{r_n} C(B_q^{2n}) \xrightarrow{\partial_n} C(S_q^{2n-1}), \qquad C(S_q^{2n+1}) \xrightarrow{z_n \mapsto 0} C(S_q^{2n-1})$$

$$r_n : S_e \mapsto P_v$$

$$\partial_n : t^{-1}(v) \mapsto 0, P_v \mapsto 0$$

A $U(1)$-equivariant pullback diagram:

$$
\begin{array}{ccc}
 & C(S_q^{2n+1})_\bullet & \\
{\scriptstyle \partial_n \circ r_n} \swarrow & & \searrow {\scriptstyle (r_n \otimes \mathrm{id}) \circ \delta} \\
C(S_q^{2n-1})_\bullet & & C(B_q^{2n}) \otimes C(S^1)_\bullet \\
{\scriptstyle \delta} \searrow & & \swarrow {\scriptstyle \partial_n \otimes \mathrm{id}} \\
 & C(S_q^{2n-1}) \otimes C(S^1)_\bullet &
\end{array}
$$

[Arici, D'Andrea, Hajac, Tobolski, 2018]

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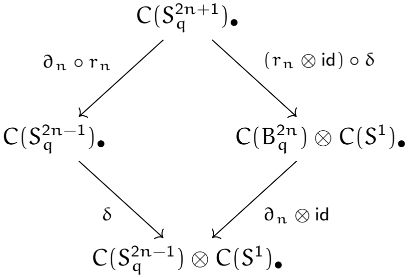

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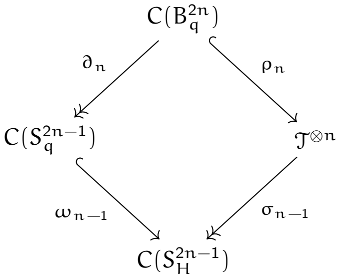
Spherical vs. non-spherical balls:



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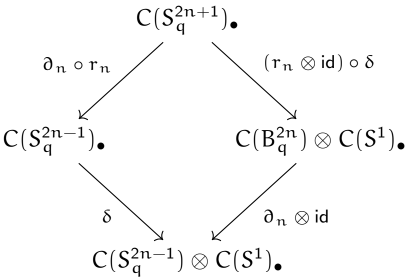

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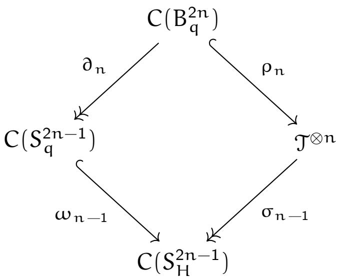


On generators:

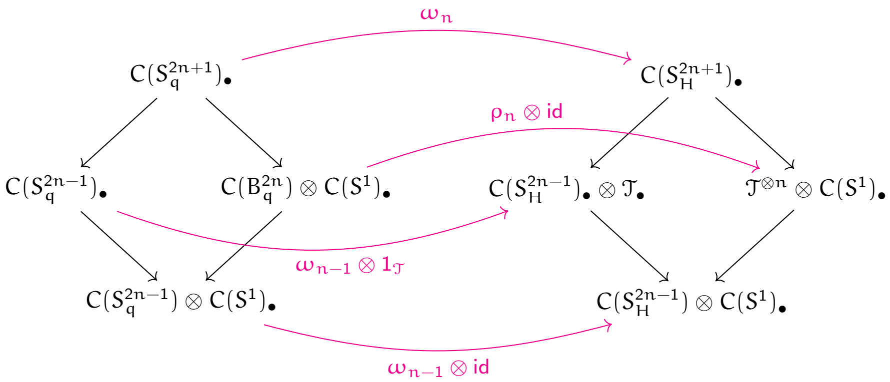
$$\rho_n(S_{e_{ij}}) := t_i t_j t_j^* \prod_{k=0}^{j-1}(1 - t_k t_k^*) \qquad \forall\, 0 \leqslant i \leqslant j < n,$$

$$\rho_n(S_{e_{in}}) := t_i \prod_{k=0}^{n-1}(1 - t_k t_k^*) \qquad \forall\, 0 \leqslant i < n,$$

$$\omega_{n-1}(S_{e_{ij}}) := s_i s_j s_j^* \prod_{k=0}^{j-1}(1 - s_k s_k^*) \qquad \forall\, 0 \leqslant i \leqslant j < n.$$

Proof of the Theorem.

With an explicit check on generators one proves the commutativity of the diagram:



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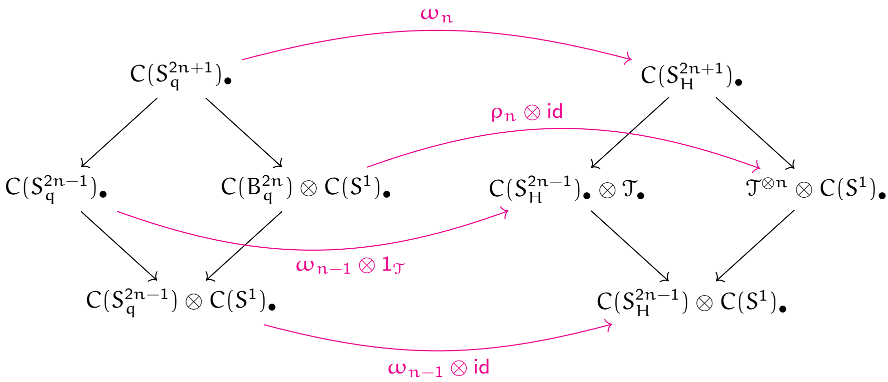

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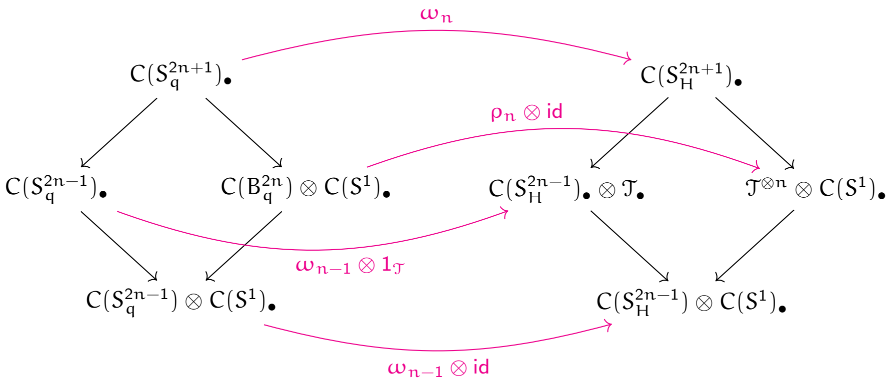

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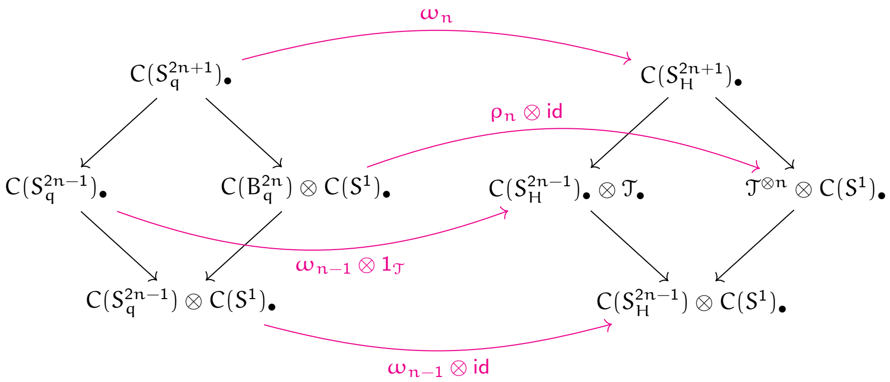

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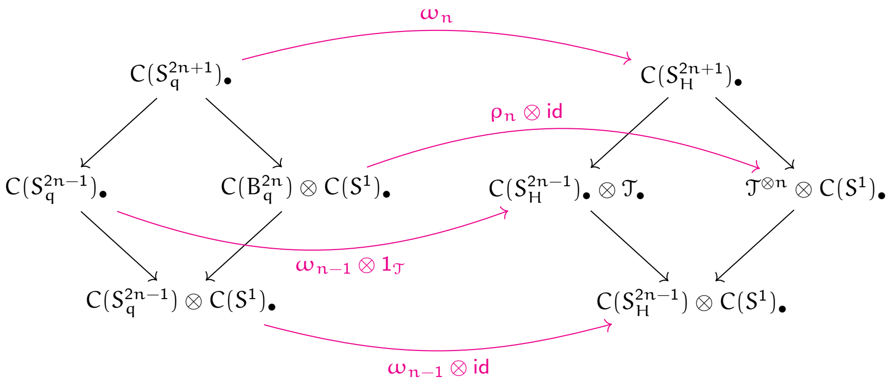

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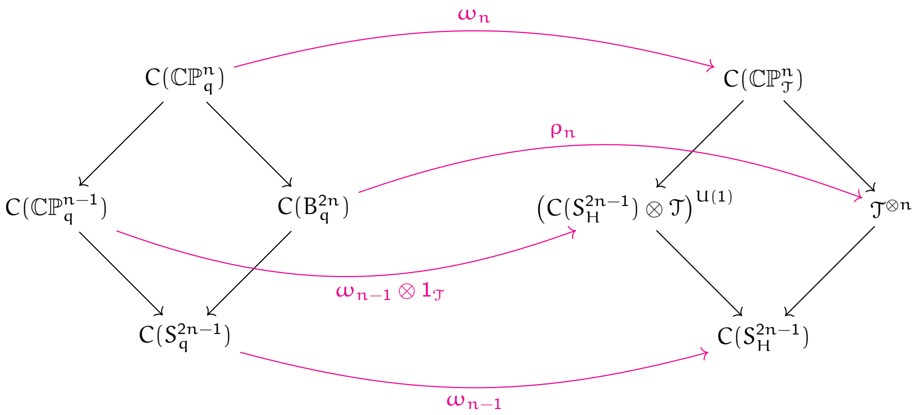
Passing to $\mathrm{U}(1)$-fixed point algebras one gets we get the commutative diagram:



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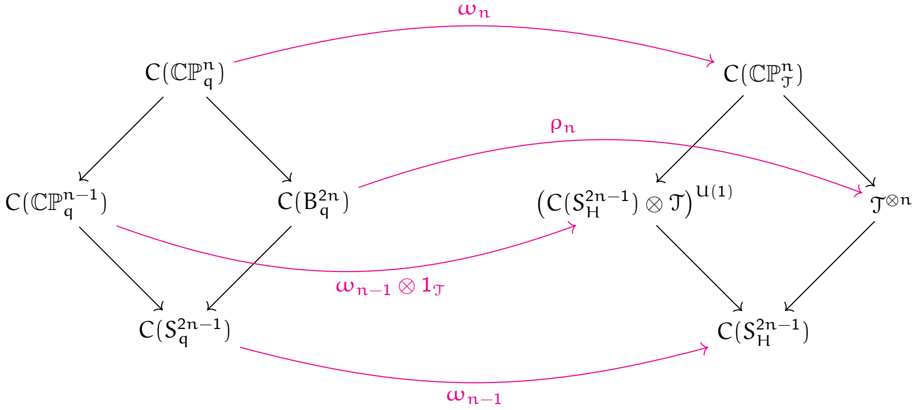

Using again Lemma 3 + induction on n we prove that the top horizontal map is a weak equivalence. ■

Questions?