Weak CW-complexes and quantum Atiyah-Todd bases

Tomasz Maszczyk

UNIVERSITY OF WARSAW

j.w. F. D'Andrea, P.M. Hajac, A. Sheu, B. Zieliński

IMPAN, 8 April 2020

Handy pre-Waldhausen categories

Our aim is to compute topological K-theory of quantum spaces admitting a structure of kind of a quantum CW-complex where we relax gluings so that we invert formally maps (*-homomorphisms of C*-algebras) inducing isomorphisms on K-theory.

Our litmus test is the case of multipullback quantum complex projective spaces, since they do not admit a CW-complex structure in a more restrictive sense, for which we construct a formalism allowing us to mimick the standard CW-complex structure of complex projective spaces.

The use of the Meyer-Vietoris principle relies on a specific structure, which we call *(handy) pre-Waldhausen,* on the category of compact quantum spaces (meaning the opposite category of unital C*-algebras).

Since general noncommutative C*-algebras rarely admit characters, we are forced to consider an unpointed version of the Waldhausen structure with an initial and a terminal object, what we indicate by the prefix *pre-*.

The choice of this prefix is motivated by the fact that for such categories there is always a canonical functor into the Waldhausen category of pointed objects.

In the classical situation of the category of compact spaces, this functor is faithful and reconstructs the K-theory of compact spaces as the reduced K-theory of compact spaces with a distinguished point added as a disjoint connected component.

To allow a well defined, at least at the level of K-theory, cell decomposition we need the property of *handyness*.

In our particular case, it allows composition of *weak hyperplane embeddings* of multipullback quantum complex projective spaces into ones of higher dimension and constructing their *weak filtration by skeleta*.

Here is the categorical framework making the above composition possible.

Definition. **A** ***pre-Waldhausen category*** $\mathscr{C}$ **is a category with an initial object** $\varnothing$ **and a terminal object** $\star$**, with distinguished two classes of maps,** Cof **of** ***cofibrations*****, depicted as** $\hookrightarrow$**, and** Weq **of** ***weak equivalences*****, depicted as** $\xrightarrow{\sim}$**, such that**

(*Cof* 1) **all isomorphisms are cofibrations,**

(*Cof* 2) **for any object** X **the unique morphism** $\varnothing \to X$ **is a cofibration,**

(*Cof* 3) **if** $X \hookrightarrow Y$ **is a cofibration and** $X \longrightarrow \widetilde{X}$ **any morphism then the pushout**

$Y \longrightarrow \widetilde{X} \sqcup_X Y$ **is a cofibration,** (*Weq* 1) **all isomorphisms are weak equivalences,**

(*Weq* 2) **weak equivalences are closed under composition,**

(*Weq* 3) ***glueing for weak equivalences*: Given any commutative diagram of the form**

$$\begin{tikzcd} Z \arrow[d, "\sim"] & X \arrow[l] \arrow[r, hook] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ \widetilde{Z} & \widetilde{X} \arrow[l] \arrow[r, hook] & \widetilde{Y} \end{tikzcd}$$

in which the vertical arrows are weak equivalences and right horizontal ones are cofibrations, the induced map $Z \sqcup_X Y \to \widetilde{Z} \sqcup_{\widetilde{X}} \widetilde{Y}$ **is a weak equivalence.**

We call a pre-Waldhausen category *handy* if

(Han) for every pushout diagram

$$\begin{tikzcd} & \widetilde{\widetilde{Z}} & \\ \widetilde{Y} \arrow[ur, "\widetilde{j}"] & & \widetilde{Z} \arrow[ul, "\widetilde{h}"'] \\ & Y \arrow[ul, "g"] \arrow[ur, "j"'] & \end{tikzcd}$$

with j being a cofibration and g being a weak equivalence, $\widetilde{h}$ is a weak equivalence as well.

Note that then, by (Cof 3), the arrow $\widetilde{j}$ is necessarilly a cofibration. Introducing the condition (Han) is motivated by the fact that uncontrolled inverting weak equivalences could lead, in principle, to unwanted collapses in the homotopy category.

The latter condition prevents this and allows one to work within the calculus of left fractions of the form $Weq^{-1} \circ Cof$ in the homotopy category

$$Ho(\mathscr{C}) := \mathscr{C}[Weq^{-1}].$$

We call the morphisms in the class $Weq^{-1} \circ Cof$ in $Ho(\mathscr{C})$ *weak cofibrations*.

Under the handyness assumption we can represent them as cospans

$$\begin{tikzcd} & \widetilde{Y} & \\ X \arrow[ur, hook] & & Y \arrow[ul, "\sim"'] \end{tikzcd}$$

denote by

$$X \rightarrowtail Y$$

and compose them in the homotopy category $Ho(\mathscr{C})$ as follows

$$\begin{tikzcd}
 & \widetilde{Z} & & & & \widetilde{Y} & & & & \widetilde{\widetilde{Z}} & \\
Y \arrow[ur, hook, "j"] & & Z \arrow[ul, "h"'] & \circ & X \arrow[ur, hook, "i"] & & Y \arrow[ul, "g"'] & = & X \arrow[ur, hook, "\widetilde{j} \circ i"] & & Z \arrow[ul, "\widetilde{g} \circ g"']
\end{tikzcd}$$

Fig. 1. Composition of weak cofibrations in $Ho(\mathscr{C})$:

$$(g^{-1} \circ i) \circ (h^{-1} \circ j) = (\widetilde{h} \circ h)^{-1} \circ (\widetilde{j} \circ i)$$

where $\widetilde{j}$ and $\widetilde{h}$ are the arrows completing the pushout square in the diagram below

$$\begin{tikzcd}
& & \widetilde{\widetilde{Z}} & & \\
& \widetilde{Y} \arrow[ur, hook, dashed, "\widetilde{j}"] & & \widetilde{Z} \arrow[ul, dashed, "\widetilde{h}"'] & \\
X \arrow[ur, hook, "i"] & & Y \arrow[ul, "g"'] \arrow[ur, hook, "j"] & & Z \arrow[ul, "h"']
\end{tikzcd}$$

Theorem 1. The opposite category of unital C*-algebras with *-homomorphisms as morphisms, zero C*-algebra as an initial object, complex numbers as a terminal object, surjective *-homomorphisms as cofibrations and *-homomorphisms inducing an isomorphism on K-theory as weak equivalences is a handy pre-Waldhausen category.

Proof. It is obvious that the zero algebra (resp. complex numbers) is a terminal (resp. initial) object in the category of unital C*-algebras.

(*Cof* 1) Every *-isomorphism of unital C*-algebras is surjective.

(*Cof* 2) The *-homomorphism into the zero algebra is surjective.

(*Cof* 3) If $\widetilde{C} \longrightarrow B$ is a surjective *-homomorphism and $\widetilde{B} \longrightarrow B$ any *-homomorphism then the pullback *-homomorphism $\widetilde{\widetilde{C}} := \tilde{B} \times_B \tilde{C} \longrightarrow \tilde{B}$ is surjective.

(*Weq* 1) and (*Weq* 2) are obvious.

(*Weq* 3) After inverting directions of all arrows, this is verbatim Thm. 3.1 of [Farsi-Hajac-Maszczyk-Zieliński '18].

(Han) **Assume that in the following pullback diagram of C*-algebras**

$$\begin{tikzcd}[column sep=small, row sep=small] & C \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & \\ A \arrow[dr, "\delta"'] & & B \arrow[dl, "\tau"] \\ & D & \end{tikzcd}$$

τ **is a surjective. Then** π **is surjective as well and** β **is a** K**-equivalence if and only if** δ **is such.**

Since surjective *-homomorphisms are regular epimorphisms, they are stable under all pullbacks. This proves surjectivity of π. Thanks to surjectivity of τ the Mayer-Vietoris theorem provides the six-term exact sequence

$$\begin{tikzcd}
K_0(C) \arrow[r, "\begin{pmatrix}\pi_* \\ \beta_*\end{pmatrix}"] & K_0(A) \oplus K_0(B) \arrow[r, "(\delta_*, -\tau_*)"] & K_0(D) \arrow[d] \\
K_1(D) \arrow[u] & K_1(A) \oplus K_1(B) \arrow[l, "(\delta_*, -\tau_*)"'] & K_1(C). \arrow[l, "\begin{pmatrix}\pi_* \\ \beta_*\end{pmatrix}"']
\end{tikzcd}$$

First we assume that δ is a K-equivalence. This implies that we have a section of the map $(\delta_*, -\tau_*)$, provided by composing $i_A : K_*(A) \to K_*(A) \oplus K_*(B)$, the standard embedding, with δ_*^{-1}. This cuts the six-term exact sequence into split short exact ones

$$\begin{tikzcd}
0 \arrow[r] & K_*(C) \arrow[r, "\binom{\pi_*}{\beta_*}"] & K_*(A) \oplus K_*(B) \arrow[r, "(\delta_*, -\tau_*)"] & K_*(D) \arrow[l, dashed, bend left, "i_A \circ \delta_*^{-1}"] \arrow[r] & 0.
\end{tikzcd}$$

This splitting produces idempotent endomorphisms of $K_*(A) \oplus K_*(B)$

$$p := i_A \circ \delta_*^{-1} \circ (\delta_*, -\tau_*) = p \circ p, \quad p^{\perp} := \mathrm{id} - p$$

such that

$$\mathrm{im}\left(\begin{smallmatrix} \pi_* \\ \beta_* \end{smallmatrix}\right) = \ker(\delta_*, -\tau_*) = \ker p \cong \mathrm{coim}\, p^{\perp} = \mathrm{coim}\, p_B$$

where p_B is a canonical projection from $K_*(A) \oplus K_*(B)$ onto $K_*(B)$.

Since both arrows in the sequence

$$K_*(C) \xrightarrow{\binom{\pi_*}{\beta_*}} \operatorname{im} \binom{\pi_*}{\beta_*} \cong \operatorname{coim} p_B \xrightarrow{p_B} K_*(B)$$

are isomorphisms and

$$p_B \circ \begin{pmatrix} \pi_* \\ \beta_* \end{pmatrix} = \beta_*,$$

β_* **is an isomorphism as well, hence** β **is a weak equivalence.**

Now, we assume that β is a K-equivalence. This implies that we have a map provided by composing β_*^{-1} with the standard projection $p_B : K_*(A) \oplus K_*(B) \to K_*(B)$, for which the map $(\delta_*, -\tau_*)$ is a section. This cuts the six-term exact sequence into split short exact ones

$$
\begin{tikzcd}
0 \arrow[r] & K_*(C) \arrow[r, "\begin{pmatrix}\pi_* \\ \beta_*\end{pmatrix}"] & K_*(A) \oplus K_*(B) \arrow[r, "(\delta_*, -\tau_*)"] \arrow[l, dashed, bend left, "\beta_*^{-1} \circ p_B"] & K_*(D) \arrow[r] & 0.
\end{tikzcd}
$$

This splitting produces idempotent endomorphisms of $K_*(A) \oplus K_*(B)$

$$q := \begin{pmatrix} \pi_* \\ \beta_* \end{pmatrix} \circ \beta_*^{-1} \circ p_B = q \circ q, \quad q^\perp := \mathrm{id} - q$$

such that

$$\mathrm{im}\, i_A = \mathrm{im}\, q^\perp \cong \mathrm{coker}\, q = \mathrm{coker} \begin{pmatrix} \pi_* \\ \beta_* \end{pmatrix} = \mathrm{coim}\,(\delta_*, -\tau_*)\,.$$

Since both arrows in the sequence

$$K_*(A) \xrightarrow{\quad i_A \quad} \operatorname{im} i_A \cong \operatorname{coim}(\delta_*, -\tau_*) \xrightarrow{(\delta_*, -\tau_*)} K_*(D)$$

are isomorphisms and

$$(\delta_*, -\tau_*) \circ i_A = \delta_*,$$

δ_* **is an isomorphism as well, hence** δ **is a weak equivalence.** $\square$

<u>Remark.</u> **Keeping in mind opposite directions of all arrows in the opposite category, the composition of weak cofibrations in the homotopy category** $Ho(C^*\text{-}Alg_1)^{op}$**, understood as generalized maps from the left to the right, reads as follows**

$$\begin{tikzcd} & \widetilde{B} \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & \\ A & & B \end{tikzcd} \circ \begin{tikzcd} & \widetilde{C} \arrow[dl, "\rho"'] \arrow[dr, "\gamma"] & \\ B & & C \end{tikzcd} = \begin{tikzcd} & \widetilde{\widetilde{C}} \arrow[dl, "\pi \circ \widetilde{\rho}"'] \arrow[dr, "\gamma \circ \widetilde{\gamma}"] & \\ A & & C \end{tikzcd}$$

Fig. 2. Composition of weak cofibrations in $Ho(C^*\text{-}Alg_1)^{op}$**:**

$$(\pi \circ \beta^{-1}) \circ (\rho \circ \gamma^{-1}) = (\pi \circ \widetilde{\rho}) \circ (\gamma \circ \widetilde{\gamma})^{-1}$$

where $\widetilde{\rho}$ and $\widetilde{\gamma}$ are the arrows completing the pullback square in the diagram below

$$\begin{tikzcd}
 & & \widetilde{\widetilde{C}} \arrow[dl, dashed, "\widetilde{\rho}"'] \arrow[dr, dashed, "\widetilde{\gamma}"] & & \\
 & \widetilde{B} \arrow[dl, "\pi"'] \arrow[dr, "\beta"] & & \widetilde{C} \arrow[dl, "\rho"'] \arrow[dr, "\gamma"] & \\
A & & B & & C
\end{tikzcd}$$

Weak CW-complex structure of $\mathbb{CP}^n_{\mathcal{T}}$. Consider the two pushout squares

$$\begin{tikzcd}
 & & \widetilde{\mathbb{CP}^n_{\mathcal{T}}} & & \\
 & \mathbb{CP}^{n-1}_{\mathcal{T}} \arrow[ur, "\widetilde{\iota}"] & & \mathbb{CP}^n_{\mathcal{T}} \arrow[ul, "\psi"'] & \\
 & & \operatorname{Tub}_{\mathbb{CP}^n_{\mathcal{T}}}(\mathbb{CP}^{n-1}_{\mathcal{T}}) \arrow[ul, "\pi"] \arrow[ur, "\epsilon"'] & & \mathbb{D}^n_{\mathcal{T}} \arrow[ul, "\phi"'] \\
 & & & S^{2n-1}_H \arrow[ul, "\chi"] \arrow[ur, "\partial"'] &
\end{tikzcd}$$

where the upper one is defined as follow.

Since the collapse $\mathrm{Tub}_{\mathbb{CP}^n_{\mathcal{T}}}(\mathbb{CP}^{n-1}_{\mathcal{T}}) \longrightarrow \mathbb{CP}^{n-1}_{\mathcal{T}}$ is a weak equivalence and by the lower pushout ϵ is a cofibration, by Theorem 1 (Han) ψ is a weak equivalence as well.

The quantum space $\widetilde{\mathbb{CP}^n_{\mathcal{T}}}$ is then the result of collapsing $\mathrm{Tub}_{\mathbb{CP}^n_{\mathcal{T}}}(\mathbb{CP}^{n-1}_{\mathcal{T}})$, the tubular neighborhood of a hyperplane $\mathbb{CP}^{n-1}_{\mathcal{T}}$ in $\mathbb{CP}^n_{\mathcal{T}}$, to this hyperplane.

Since the concatenation of pushout squares is a pushot square, we obtain a new pushout square, the outer one

$$\begin{tikzcd}
 & \widetilde{\mathbb{CP}^n_{\mathcal{T}}} & \\
\mathbb{CP}^{n-1}_{\mathcal{T}} \arrow[ur, "\widetilde{\iota}"] & & \mathbb{D}^n_{\mathcal{T}} \arrow[ul, "\widetilde{\phi}"'] \\
 & S^{2n-1}_H \arrow[ul, "h"] \arrow[ur, "\partial"'] &
\end{tikzcd}$$

where $h := \pi \circ \chi$ is a quantum Hopf fibration and $\widetilde{\phi} := \psi \circ \phi$, accompanied by a weak equivalence

$$\psi : \mathbb{CP}^n_{\mathcal{T}} \xrightarrow{\sim} \widetilde{\mathbb{CP}^n_{\mathcal{T}}}.$$

Note that since ψ is a weak equivalence, the upper cospan defines a weak cofibration

$$\iota := \psi^{-1} \circ \widetilde{\iota} : \mathbb{CP}^{n-1}_{\mathcal{T}} \rightarrowtail \mathbb{CP}^{n}_{\mathcal{T}}$$

in its homotopy category. We can understood it as a weak replacement of the classical *hyperplane embedding*.

Therefore, since we now can compose weak cofibrations, we can form a *weak filtration by skeleta* where at every step we attach a single quantum cell, as in the classical case

$$\mathbb{CP}^0_{\mathcal{T}} \rightarrowtail \mathbb{CP}^1_{\mathcal{T}} \rightarrowtail \cdots \rightarrowtail \mathbb{CP}^{n-1}_{\mathcal{T}} \rightarrowtail \mathbb{CP}^n_{\mathcal{T}}.$$

It induces a system of $\mathbb{Z}/2\mathbb{Z}$-graded K-groups

$$0 \leftarrow K^*(\mathbb{CP}^0_{\mathcal{T}}) \leftarrow K^*(\mathbb{CP}^1_{\mathcal{T}}) \leftarrow \cdots \leftarrow K^*(\mathbb{CP}^{n-1}_{\mathcal{T}}) \leftarrow K^*(\mathbb{CP}^n_{\mathcal{T}}).$$

Quantum Atiyah-Todd picture

The classical case revisited. The classical result of Atiyah-Todd says that $K^0(\mathbb{CP}^n)$ equipped with the ring structure defined via the tensor product of vector bundles over $\mathbb{CP}^n$ fits into the following commutative square of rings:

$$\begin{array}{ccc} \mathbb{Z}[t, t^{-1}] & \xrightarrow{\cong} & R(U(1)) \\ \downarrow & & \downarrow \\ \mathbb{Z}[x]/(x^{n+1}) & \xrightarrow{\cong} & K^0(\mathbb{CP}^n). \end{array}$$

Here the left vertical arrow is given by $t \mapsto 1 + x$, the right vertical arrow is induced by the associated vector bundle construction, the top isomorphism maps t into the fundamental representation of $U(1)$ in the representation ring $R(U(1))$, and the bottom isomorphism maps x to the K-theory element $[\mathrm{L}_1] - [1]$, where L_1 denotes the Hopf line bundle on $\mathbb{CP}^n$ associated with the fundamental representation of $U(1)$.

Below, for any $k \in \mathbb{Z}$, we denote by L_k the k-th tensor power of L_1 when k is non-negative, and the $|k|$-th tensor power of L_{-1} when k is negative, where L_{-1} is the Hopf line bundle on $\mathbb{CP}^n$ associated with the dual of the fundamental representation of $U(1)$.

Equivalently, L_k is the Hopf line bundle on $\mathbb{CP}^n$ associated with the k-th tensor power of the fundamental representation of $U(1)$, where negative tensor powers refer to tensor powers of the dual of the fundamental representation of $U(1)$.

Since the elements $(1+x)^k$, $k = 0, \dots, n$, form a basis of the free $\mathbb{Z}$-module $\mathbb{Z}[x]/(x^{n+1})$ and the assignment $(1+x) \mapsto [\mathrm{L}_1]$ gives an isomorphism of rings, the classes

$$[\mathrm{L}_0], \dots, [\mathrm{L}_n]$$

form the *Atiyah-Todd basis* of $K^0(\mathbb{CP}^n)$.

Our next step is to unravel how the classes $[\mathrm{L}_k]$**, for** $k = -1$ **or** $k = n + 1$**, can be expressed in the Atiyah-Todd basis. Note first that the equality**

$$0 = x^{n+1} = ((1 + x) - 1)^{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} (1 + x)^k$$

in $\mathbb{Z}[x]/(x^{n+1})$ **translates to the equality in** $K^0(\mathbb{CP}^n)$

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} [\mathrm{L}_k] = 0.$$

Thus we obtain

$$[\mathrm{L}_{n+1}] = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} [\mathrm{L}_k],$$

which we will refer to as the *first Atiyah-Todd identity*.

Furthermore, since $(1+x)$ is invertible in $\mathbb{Z}[x]/(x^{n+1})$ and the initial equality can be rewritten as

$$(1+x)\sum_{k=1}^{n+1}(-1)^{1-k}\binom{n+1}{k}(1+x)^{k-1}=1,$$

we obtain

$$(1+x)^{-1}=\sum_{k=1}^{n+1}(-1)^{1-k}\binom{n+1}{k}(1+x)^{k-1}=\sum_{k=0}^{n}(-1)^{k}\binom{n+1}{k+1}(1+x)^{k}$$

in $\mathbb{Z}[x]/(x^{n+1})$.

This equality translates to $K^0(\mathbb{CP}^n)$ as

$$[L_{-1}] = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} [L_k].$$

We will refer to it as the *second Atiyah-Todd identity*.

In Prop. 3.3 and 3.4 of [Arici-Brain-Landi ‘15], the additive version of the bottom isomorphism in the Atiyah-Todd diagram was established for the Vaksman-Soibelman quantum complex projective spaces $\mathbb{CP}_q^n$. It yields a noncommutative version of the Atiyah-Todd basis for $\mathbb{CP}_q^n$.

All this seems interesting because Atiyah-Todd's method to prove the existence of the Atiyah-Todd diagram uses the ring structure of K-theory, which is missing in the noncommutative setting. Instead, the index pairing is used which is merely additive.

Below we not only obtain an analog of the Atiyah-Todd basis for $\mathbb{CP}^n_{\mathcal{T}}$, but also we establish analogues of the Atiyah-Todd identities and which are lacking in Prop. 3.3 and 3.4 of [Arici-Brain-Landi '15].

The multipullback noncommutative deformation.

Although the K_0-group of a noncommutative C*-algebra does not have an intrinsic ring structure, it turns out that, much as in the Atiyah-Todd diagram, the abelian group $K_0(C(\mathbb{CP}^n_{\mathcal{T}}))$ is a free module of rank one over the ring $\mathbb{Z}(x)/(x^n)$.

The basis of this free module is the K_0-class of $C(\mathbb{CP}^n_{\mathcal{T}})$. The module structure comes from tensoring finitely generated projective $C(\mathbb{CP}^n_{\mathcal{T}})$-modules by the bimodules associated with the quantum Hopf $U(1)$-principal bundle $S^{2n+1}_H \to \mathbb{CP}^n_{\mathcal{T}}$.

Moreover, we will show that, despite the aforementioned lack of an intrinsic ring structure, we still enjoy analogs of the Atiyah-Todd identities.

Recall that, we denote by S_H^{2n+1} the multipullback $(2n+1)$-dimensional quantum sphere [Hajac-Nest-Pask-Sims-Zieliński '18] and by $\mathbb{CP}_{\mathcal{T}}^n$ the corresponding multipullback quantum complex projective space [Hajac-Kaygun-Zieliński '12], whose C*-algebra we identify with a $U(1)$-fixed-point subalgebra of $C(S_H^{2n+1})$ [Hajac-Nest-Pask-Sims-Zieliński '18].

Next, let

$$\partial_{n+1} : \mathcal{T}^{\otimes(n+1)} \longrightarrow \mathcal{T}^{\otimes(n+1)}/\mathcal{K}^{\otimes(n+1)} \cong C\left(S_H^{2n+1}\right)$$

be the canonical quotient map from Lemma 5.1 of [Hajac-Nest-Pask-Sims-Zieliński '18], and let

$$P_k := \sum_{i=1}^{k} e_{ii} \in \mathcal{K} \subset \mathcal{T}, \quad P_k^{\perp} := I - P_k \in \mathcal{K}^{+} \subset \mathcal{T}, \quad k \in \mathbb{N}.$$

Here e_{ij} **with** $i, j \in \mathbb{N}$ **represents a matrix unit in** $\mathcal{K}$ **which we identify with** $\mathcal{K}\left(\ell^2\left(\mathbb{N}\right)\right)$**, and** $\mathcal{K}^{+}$ **stands for the minimal unitization of** $\mathcal{K}$**.**

Note that, according to the standard summation-over-the-empty-set convention, $P_0 := 0$, so $P_0^{\perp} = I$. For finite square matrices $P, Q \in M_{\infty}(A)$ with entries in a unital C*-algebra A, we use the notion $P \sim_A Q$ to denote that they are unitarily equivalent over A, and use $P \boxplus Q$ to denote their diagonal direct sum.

Furthermore, for $0 \leq j \leq n$ and $k \geq 0$, we define the projections

$$E_k^j := \partial_{n+1}\big((\otimes^j P_1) \otimes P_k^{\perp} \otimes (\otimes^{n-j} I)\big) \in C(\mathbb{CP}^n_{\mathcal{T}}).$$

Note that $E_k^n = \partial_{n+1}((\otimes^n P_1) \otimes P_k^{\perp}) = \partial_{n+1}((\otimes^n P_1) \otimes I)$ since $\partial_{n+1}((\otimes^n P_1) \otimes P_k) = 0$. In particular,

$$E_k^n = E_{k+1}^n$$

is independent of k.

For the sake of forthcoming recursive formulas, we adopt the convention $E_k^{n+1} := 0$ and $0^0 := 1$.

Now, recall from Theorem 4 [Sheu '19] and the remark below this theorem that, for $j = 0, \ldots, n$, the classes $[E_0^j]$ form a basis of the free $\mathbb{Z}$-module $K_0(C(\mathbb{CP}^n_{\mathcal{T}})) \cong \mathbb{Z}^{n+1}$.

Next, remembering that $E_k^j \in C(\mathbb{CP}^n_{\mathcal{T}})$ **(they are all** $U(1)$**-invariant), we will follow an argument used in [Sheu '19] to establish** $\left\{[\partial_n((\otimes^j I) \otimes (\otimes^{n-j} P_1))]\right\}_{0<j\leq n}$ **as a basis of** $K_0(C(\mathbb{CP}^{n-1}_{\mathcal{T}}))$**, to prove the recursive relation in** $K_0(C(\mathbb{CP}^n_{\mathcal{T}}))$

$$[E_{k+1}^j] = [E_k^j] - [E_k^{j+1}].$$

To this end, we need the following lemma:

Lemma 1. **Let S be the generating isometry of the Toeplitz algebra $\mathcal{T}$ identified with the unilateral shift on the Hilbert space $\ell^2(\mathbb{N})$. For any $k \geq 0$ and $n \geq 1$,**

$$u_k := \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} \in M_2(\mathcal{T}^{\otimes 2})$$

is a self-adjoint unitary conjugating $(e_{kk} \otimes I) \boxplus 0$ to $0 \boxplus (P_1 \otimes P_k^{\perp})$.

Proof. **First, we verify that the self-adjoint element $u_k \in M_2(\mathcal{T}^{\otimes 2})$ is unitary:**

$$\begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix}$$

$$= \begin{pmatrix} P_k \otimes I + S^k(S^k)^* \otimes (S^k)^* S^k & P_k S^k \otimes (S^k)^* + S^k \otimes (S^k)^* P_k \\ (S^k)^* P_k \otimes S^k + (S^k)^* \otimes P_k S^k & (S^k)^* S^k \otimes S^k (S^k)^* + I \otimes P_k \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} P_k \otimes I + P_k^{\perp} \otimes I & 0 \otimes (S^k)^* + S^k \otimes 0 \\ 0 \otimes S^k + (S^k)^* \otimes 0 & I \otimes P_k^{\perp} + I \otimes P_k \end{pmatrix} \\
&= \begin{pmatrix} I \otimes I & 0 \\ 0 & I \otimes I \end{pmatrix}.
\end{aligned}
$$

Next, u_k **conjugates** $(e_{kk} \otimes I) \boxplus 0$ **to** $0 \boxplus \left(P_1 \otimes P_k^{\perp}\right)$ **because**

$$\begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} \begin{pmatrix} e_{kk} \otimes I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ e_{0k} \otimes S^k & 0 \end{pmatrix} \begin{pmatrix} P_k \otimes I & S^k \otimes (S^k)^* \\ (S^k)^* \otimes S^k & I \otimes P_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & e_{00} \otimes P_k^{\perp} \end{pmatrix}.$$

□

Lemma 1. **For any** $0 \le j \le n$ **and any** $k \ge 0$**,**

$$[E^j_{k+1}] = [E^j_k] - [E^{j+1}_k].$$

Proof. **First, note that the statements are true for** $j = n$ **because** $E^n_k = E^n_{k+1}$ **is independent of** k**, and** $E^{n+1}_k := 0$**. Hence, we can assume** $0 \le j < n$**.**

Furthermore, since $P_k^\perp = P_{k+1}^\perp + e_{kk}$ **and the summands are orthogonal projections, we obtain**

$$E_k^j = \partial_{n+1}((\otimes^j P_1) \otimes P_k^\perp \otimes (\otimes^{n-j} I))$$

$$\sim_{C(\mathbb{CP}^n_{\mathcal{T}})} \partial_{n+1}((\otimes^j P_1) \otimes P_{k+1}^\perp \otimes (\otimes^{n-j} I)) \boxplus \partial_{n+1}((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I))$$

$$= E_{k+1}^j \boxplus \partial_{n+1}((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I)).$$

Therefore, to finish the proof, it suffices to show the following auxiliary identity

$$[\partial_{n+1}((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I))] = [E_k^{j+1}].$$

To this end, we take advantage of Lemma 1 to conclude that $(\otimes^j P_1) \otimes u_k \otimes (\otimes^{n-j-1} I)$ **conjugates** $((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I)) \boxplus 0$ **to**

$$0 \boxplus ((\otimes^j P_1) \otimes P_1 \otimes P_k^{\perp} \otimes (\otimes^{n-j-1} I)) = 0 \boxplus E_k^{j+1} .$$

Here the tensor product $(\otimes^j P_1) \otimes u_k \otimes (\otimes^{n-j-1} I)$ **is understood entrywise with respect to the matrix** u_k**.**

Finally, since $\partial_{n+1}(a_{ij})$ is $U(1)$-invariant for each entry a_{ij} of $(\otimes^j P_1) \otimes u_k \otimes (\otimes^{n-j-1} I)$, we have $\partial_{n+1}(a_{ij}) \in C(\mathbb{CP}^n_{\mathcal{T}})$, so

$$\partial_{n+1}(((\otimes^j P_1) \otimes e_{kk} \otimes (\otimes^{n-j} I)) \boxplus 0) \ \sim_{C(\mathbb{CP}^n_{\mathcal{T}})} \ 0 \boxplus E_k^{j+1}.$$

Passing to the K_0-classes, we obtain the desired auxiliary identity. $\square$

Having shown the recursive relation, we are ready to prove

Lemma 2. **For any** $k \geq 0$**,**

$$[L_k] = \sum_{j=0}^{k} (-1)^j \binom{k}{j} [E_0^j].$$

Proof. By Theorem 6 of [Sheu '19], for $k \geq 0$, the modules L_k are represented, respectively, by the projections $\partial_{n+1}\left(P_k^{\perp} \otimes (\otimes^n I)\right) =: E_k^0$.

Starting from $l = 0$, we prove inductively, for $0 \leq l \leq k$ with $k \geq 0$ fixed, the intermediate identity

$$[L_k] = \sum_{j=0}^{l} (-1)^j \binom{l}{j} [E_{k-l}^j].$$

This equation is clearly true for $l = 0$. Now, for $0 < l \leq k$, taking advantage of the induction hypothesis and the recursive relation as in Lemma 1, we compute:

$$\begin{aligned}
[L_k] &= \sum_{j=0}^{l-1} (-1)^j \binom{l-1}{j} [E^j_{k-l+1}] \\
&= \sum_{j=0}^{l-1} (-1)^j \binom{l-1}{j} \left([E^j_{k-l}] - [E^{j+1}_{k-l}] \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{l-1} \left((-1)^j \binom{l-1}{j} [E_{k-l}^j] + (-1)^{j+1} \binom{l-1}{j} [E_{k-l}^{j+1}] \right) \\
&= [E_{k-l}^0] + \sum_{j=1}^{l-1} (-1)^j \left(\binom{l-1}{j} [E_{k-l}^j] + \binom{l-1}{j-1} [E_{k-l}^j] \right) + (-1)^l \binom{l-1}{l-1} [E_{k-l}^l]
\end{aligned}
$$

$$= [E^0_{k-l}] + \sum_{j=1}^{l-1} (-1)^j \binom{l}{j} [E^j_{k-l}] + (-1)^l [E^l_{k-l}]$$

$$= \sum_{j=0}^{l} (-1)^j \binom{l}{j} [E^j_{k-l}].$$

This proves the intermediate identity, which, for $l = k$, becomes the desired identity. $\square$

Theorem 2. For any $n \in \mathbb{N}$, we have **quantum Atiyah-Todd basis and identities**:

$$K_0(C(\mathbb{CP}^n_{\mathcal{T}})) = \bigoplus_{k=0}^{n} \mathbb{Z}[L_k],$$

$$[L_{n+1}] = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} [L_k],$$

$$[L_{-1}] = \sum_{k=0}^{n} (-1)^{k} \binom{n+1}{k+1} [L_k].$$

Proof. To begin with, note that the first equality follows immediately from Lemma 2 and Theorem 4 of [Sheu '19] because the expansion coefficients $(-1)^j \binom{k}{j}$ in Lemma 2 form a matrix in $GL_{n+1}(\mathbb{Z})$. (The matrix is lower-triangular of determinant ± 1.)

Next, to prove the first quantum Atiyah-Todd identity, we show an equivalent identity:

$$\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^{n+1-k}\tbinom{n+1}{k}[L_k] \\
&=\sum_{k=0}^{n+1}(-1)^{n+1-k}\tbinom{n+1}{k}\left(\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}[E_0^j]\right) \\
&=\sum_{j=0}^{n+1}\sum_{k=j}^{n+1}(-1)^{n+1+j-k}\tbinom{n+1}{k}\tbinom{k}{j}[E_0^j]
\end{aligned}$$

$$= \sum_{j=0}^{n+1} \left(\sum_{k=j}^{n+1} (-1)^{n+1+j-k} \frac{(n+1)!}{k!(n+1-k)!} \frac{k!}{j!(k-j)!} \right) [E_0^j]$$

$$= \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} \left(\sum_{k=j}^{n+1} (-1)^{n+1+j-k} \frac{(n+1-j)!}{(n+1-k)!(k-j)!} \right) [E_0^j]$$

$$
\begin{aligned}
&= \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!}(-1)^j \left(\sum_{k=0}^{n+1-j} (-1)^{n+1-j-k} \frac{(n+1-j)!}{(n+1-j-k)!k!} \right) [E_0^j] \\
&= \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!}(-1)^j (1+(-1))^{n+1-j} [E_0^j] = 0.
\end{aligned}
$$

Finally, to prove the second quantum Atiyah-Todd identity, we recall from [Sheu '19] that the class $[L_{-1}]$ can be represented by the projection $\boxplus_{j=0}^{n} E_0^j$. Thus the second quantum Atiyah-Todd identity becomes

$$\sum_{k=0}^{n}(-1)^k \binom{n+1}{k+1} [L_k] = \sum_{j=0}^{n}[E_0^j].$$

The left-hand-side can be computed as follows:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \binom{n+1}{k+1} [L_k] &= \sum_{k=0}^{n}(-1)^k \binom{n+1}{k+1} \left(\sum_{j=0}^{k}(-1)^j \binom{k}{j} [E_0^j]\right) \\ &= \sum_{j=0}^{n}\sum_{k=j}^{n} \frac{(-1)^{k+j}(n+1)!}{(k+1)!(n-k)!}\frac{k!}{j!(k-j)!}[E_0^j]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \frac{(n+1)!}{j!(n-j)!} \left(\sum_{k=j}^{n} \frac{(-1)^{k+j}(n-j)!}{(n-k)!(k-j)!} \frac{1}{k+1} \right) [E_0^j] \\
&= \sum_{j=0}^{n} \frac{(n+1)!}{j!(n-j)!} \left(\sum_{k=0}^{n-j} \frac{(-1)^{k}(n-j)!}{(n-j-k)!k!} \frac{1}{k+j+1} \right) [E_0^j] \\
&= \sum_{j=0}^{n} \frac{(n+1)!}{j!(n-j)!} \left(\sum_{k=0}^{n-j} (-1)^{k} \binom{n-j}{k} \frac{1}{k+j+1} \right) [E_0^j].
\end{aligned}
$$

Hence it remains to show that, for all $0 \leq j \leq n$,

$$\frac{(n+1)!}{j!(n-j)!}\left(\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{1}{k+j+1}\right)=1.$$

To this end, we introduce auxiliary polynomials over $\mathbb{Q}$:

$$f_j(x) := \sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{1}{k+j+1}x^{k+j+1},$$

which can be evaluated and formally differentiated and integrated.

Now our goal can be rephrased as follows:

$$\frac{j!(n-j)!}{(n+1)!} = f_j(1).$$

To compute this, note first that

$$f_j'(x) = \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} x^{k+j} = (-1)^{n-j} x^j (x-1)^{n-j}.$$

Therefore, as $f_j(0) = 0$ because $k, j \geq 0$, we obtain:

$$\begin{aligned} f_j(1) &= \int_0^1 (-1)^{n-j} x^j (x-1)^{n-j} dx \\ &= \frac{(-1)^{n-j}}{j+1} x^{j+1} (x-1)^{n-j} \Big|_0^1 - \int_0^1 \frac{(-1)^{n-j}(n-j)}{j+1} x^{j+1} (x-1)^{n-j-1} dx \\ &= \frac{(-1)^{n+1-j}(n-j)}{j+1} \int_0^1 x^{j+1} (x-1)^{n-j-1} dx. \end{aligned}$$

Iterating this kind of integration by parts, we infer that

$$\begin{aligned} f_j(1) &= \frac{(-1)^{n+(n-j)-j}(n-j)!}{(j+1)(j+2)\cdots(j+(n-j))} \int_0^1 x^{j+(n-j)}(x-1)^0 dx \\ &= \frac{(n-j)!}{(j+1)(j+2)\cdots n} \int_0^1 x^n dx = \frac{(n-j)!}{(j+1)(j+2)\cdots n(n+1)} \\ &= \frac{j!(n-j)!}{(n+1)!}, \end{aligned}$$

as desired. $\square$

The $R(U(1))$-module structure on $K^0(\mathbb{CP}^n_{\mathcal{T}}) := K_0(C(\mathbb{CP}^n_{\mathcal{T}}))$

Assume that a free action of a compact quantum group $\mathbb{G}$ on a C*-algebra A is given, with the subalgebra $B = A^{\mathbb{G}}$ of invariants. Let $H = \mathscr{O}(\mathbb{G})$ be the the Peter-Weyl Hopf dense *-subalgebra in the C*-algebra $C(\mathbb{G})$ "of continuous functions on $\mathbb{G}$" and $\mathscr{A}$ be the Peter-Weyl dense H-comodule *-B-subalgebra in A.

Given a representation V of $\mathbb{G}$ equivalent to a finite dimensional left H-comodule V one has a finitely generated projective from either side associated B-module $A \Box^H V$. This B-bimodule defines an endofunctor $(-) \otimes_B (A \Box^H V)$ on the exact category of finitely generated projective right B-modules. Since the association is a strong monoidal functor from the category of left H-comodules to the category of B-bimodules, it defines an action of the representation ring $R(\mathbb{G})$ of $\mathbb{G}$ on the topological K-theory of B.

The action on the distinguished class $[B] \in K_*(B)$ defines a right $R(\mathbb{G})$-module map

$$R(\mathbb{G}) \to K_*(B),$$

essentially being forgetting of the left B-module structure of an associated finitely generated projective B-bimodule. In our case, when $A = C(S_H^{2n+1})$, $\mathbb{G} = U(1)$, $B = C(\mathbb{CP}_{\mathcal{T}}^n)$, we obtain a map of right $R(U(1))$-modules

$$R(U(1)) \to K^*(\mathbb{CP}_{\mathcal{T}}^n),$$

Theorem 3. **The above map of right $R(U(1))$-modules and the left-hand-side map being a ring map induced by $t \mapsto 1 + x$ fit into the following diagram of right $\mathbb{Z}[t, t^{-1}]$-modules**

$$\begin{array}{ccc} \mathbb{Z}[t, t^{-1}] & \xrightarrow{\cong} & R(U(1)) \\ \downarrow & & \downarrow \\ \mathbb{Z}[x]/(x^{n+1}) & \xrightarrow{\cong} & K^0(\mathbb{CP}^n_{\mathcal{T}}). \end{array}$$

In particular, $K^*(\mathbb{CP}^n_{\mathcal{T}})$ is a rank one free right $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}[x]/(x^{n+1})$-module, where x is even, generated by the class $[L_0]$.

Proof. By Theorem 2 the canonical right $\mathbb{Z}[t, t^{-1}]$-module structure on the free $\mathbb{Z}$-module

$$K^0(\mathbb{CP}^n_{\mathcal{T}}) = \bigoplus_{k=0}^{n} \mathbb{Z}[L_k]$$

is uniquely determined as shifting the winding number by one

$$[L_k]t = [L_{k+1}], \quad \text{for } k = 0, \ldots, n-1,$$

$$[L_n]t = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} [L_k],$$

and by minus one

$$[L_0]t^{-1} = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} [L_k],$$

$$[L_k]t^{-1} = [L_{k-1}], \qquad \text{for } k = 1, \ldots, n.$$

The fact that the minimum polynomial of the matrix of the right action of t is equal to $(t-1)^{n+1}$ and the isomorphism of rings

$$\mathbb{Z}[t,t^{-1}]/((t-1)^{n+1}) \cong \mathbb{Z}[x]/(x^{n+1})$$

$$t \mapsto 1+x, \quad t^{-1} \mapsto 1-x+x^2-\ldots+(-1)^n x^n$$

together prove the bottom isomorphism fitting into the quantum Atiyah-Todd diagram of $\mathbb{Z}[t,t^{-1}]$-modules. $\square$

Remark. **Since K-theory of a noncommutative ring lacks an intrinsic ring structure, the above $\mathbb{Z}[x]/(x^{n+1})$-module structure is best one could expect about the structure of K-theory of $\mathbb{CP}^n_{\mathcal{T}}$.**

Another good feature of this module structure is compatibility with the *weak filtration by skeleta*, where the tower of K-theories of the *weak skeleta* becomes the tower of truncated polynomials

$$\begin{tikzcd}
0 & K^*(\mathbb{CP}^0_{\mathcal{T}}) \arrow[l, two heads] \arrow[d, "\cong"] & K^*(\mathbb{CP}^1_{\mathcal{T}}) \arrow[l, two heads] \arrow[d, "\cong"] & \cdots \arrow[l, two heads] & K^*(\mathbb{CP}^{n-1}_{\mathcal{T}}) \arrow[l, two heads] \arrow[d, "\cong"] & K^*(\mathbb{CP}^n_{\mathcal{T}}) \arrow[l, two heads] \arrow[d, "\cong"] \\
0 & \mathbb{Z} \arrow[l, two heads] & \mathbb{Z}[x]/(x^2) \arrow[l, two heads] & \cdots \arrow[l, two heads] & \mathbb{Z}[x]/(x^n) \arrow[l, two heads] & \mathbb{Z}[x]/(x^{n+1}) \arrow[l, two heads]
\end{tikzcd}$$

giving the generators of the kernels of the succesive restriction morphisms of K-theory to lower weak skeleta in terms of the quantum Atiyah-Todd basis, e.g.

$$[L_0]x^n = [L_0](t-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [L_k] \in K^*(\mathbb{CP}^n_{\mathcal{T}}).$$