Multiplicative lattices, commutators, braces

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The algebra of the Yang-Baxter equation
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Thank you

Agata Smoktunowicz (two conferences: Graz 20 July 2021, Jerusalem 3 November 2021)

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9 months of exciting mathematical research

My coauthors

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Multiplicative lattices

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(No associativity, commutativity, identities, distributivity required.)

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Hence we need a lattice + a multiplication.

A non-associative multiplication!

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$$[A_3,[S_3,S_3]] = [A_3,A_3] = 1,$$

but

$$[[A_3,S_3],S_3] = A_3.$$

(1) Alberto Facchini, Carmelo Finocchiaro (Università di Catania, Italy) and George Janelidze (University of Cape Town, South Africa), "Abstractly constructed prime spectra", Algebra Universalis 83(1) (February 2022)

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(4) Dominique Bourn, Alberto Facchini and Mara Pompili, Aspects of the Category SKB of Skew Braces, submitted for publication, 2022, available in arXiv.

Our first aim: to collect under one umbrella all notions of Zariski spectra

Any (commutative or noncommutative) ring (with or without an identity), any bounded distributive lattice L, commutative semiring with identity, commutative C^*-algebra, commutative monoids, abelian ℓ-groups, MV-algebra, continuous lattice, L-algebra, left skew brace, ...$\mapsto$ a Zariski spectrum, which is a sober space.

In all the previous examples, there is a multiplicative lattice around:
For commutative rings: the lattice of its ideals with multiplication of ideals.
For noncommutative rings: the lattice of its two-sided ideal with multiplication of ideals, or $IJ + JI$ as a product, if you prefer.
For groups: the modular lattice of its normal subgroups with commutator of two normal subgroups.
For L-algebras: the lattice of all ideals with intersection of two ideals.
For lattices: the lattice itself with multiplication $xy := x \wedge y$, or the lattice of its ideals.

The notions of prime ideal, semiprime ideal, Zariski spectrum, commutator, solvable object (algebraic structure), nilpotent object, abelian object, idempotent object, hyperabelian object, Jacobson radical, centralizer, center, central series, derived series, etc., find all their natural setting in multiplicative lattices.

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Left skew braces

[Guarnieri and Vendramin] A *(left) skew brace* is a triple $(A, *, \circ)$, where $(A, *)$ and $(A, \circ)$ are groups such that

$$a \circ (b * c) = (a \circ b) * a^{-1} * (a \circ c) \tag{1}$$

for every $a, b, c \in A$. Here a^{-1} denotes the inverse of a in the group $(A, *)$.

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Left skew braces are digroups.

Ideals in skew braces

Morphisms of left skew braces (or more generally digroups) are defined in the standard way of Universal Algebra, and we get two categories SKB and DiGp. The zero object of these categories is the left skew brace (digroup) with one element.

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Ideal in a digroup $(A, *, \circ)$ = subset I of A such that I is a normal subgroup of $(A, *)$, I is a normal subgroup of $(A, \circ)$ and, for all $a, b \in A$, $a^{-1} * b \in I$ if and only if $a' \circ b \in I$ (this simply means that $a * I = a \circ J$ for every $a \in A$).

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Ideal in a left skew brace $(A, *, \circ)$ = subset I of A such that I is a normal subgroup of $(A, *)$, I is a normal subgroup of $(A, \circ)$ and $\lambda_a(I) \subseteq I$ for every $a \in A$ (again, this simply means that $a * I = a \circ J$ for every $a \in A$).

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Any intersection of ideals is an ideal, so that we get a complete lattice $\mathcal{I}(A)$ of ideals for any skew brace A. In particular, every subset X of A generates an ideal, the intersection of the ideals that contain X. It can be "constructively" described as follows. Given a subset X of a skew brace A, consider the increasing sequence X_n, $n \geq 0$, of subsets of A where $X_0 := X$, X_{n+1} is the normal closure of X_n in $(A, *)$ if $n \equiv 0 \pmod 3$, X_{n+1} is the normal closure of X_n in $(A, \circ)$ if $n \equiv 1$ (mod 3, $X_{n+1} := \bigcup_{a \in A} \lambda_a(X_n)$ if $n \equiv 2 \pmod 3$. The ideal of A generated by X is $\bigcup_{n \geq 0} X_n$.

The lattice of all ideals of a left skew brace

In the complete lattice of all ideals $\mathcal{I}(A)$ of a left skew brace $(A, *, \circ)$ one has that $I \vee J = I * J = I \circ J$, and $I \wedge J = I \cap J$. Hence the two forgetful functors $\mathsf{SKB} \to \mathsf{Gp}$, $(A, *, \circ) \to (A, *)$ and $\mathsf{SKB} \to \mathsf{Gp}$, $(A, *, \circ) \to (A, \circ)$ induce two lattice embeddings $\mathcal{I}(A) \hookrightarrow \mathcal{N}(A, *)$ and $\mathcal{I}(A) \hookrightarrow \mathcal{N}(A, \circ)$.

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Now that we have a complete lattice $\mathcal{I}(A)$, we need a multiplication on it.

Commutators

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The category of braces is semiabelian category with a zero object. If I, J are ideals of a left skew brace A, the Huq commutator $[I, J]_H$ is the smallest ideal X of A such that there is a left skew brace morphism $\varphi\colon I \times J \to A/X$ for which the diagram

$$\begin{tikzcd} I \arrow[r, "(1_I, 0_I)"] \arrow[d, hook] & I \times J \arrow[d, dotted, "\varphi"'] & J \arrow[l, "(0_J, 1_J)"'] \arrow[d, hook] \\ A \arrow[r, two heads] & A/X & A \arrow[l, two heads] \end{tikzcd}$$

commute.

Mal'tsev

[Mal'tsev, 1954]. The following conditions are equivalent for any variety of algebras $\mathcal{V}$:
(a) $\mathcal{V}$ is congruence-permutable.
(b) There is a ternary term q such that

$$\mathcal{V} \models q(x, y, y) \approx x \approx q(y, y, x).$$

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Such a term is called a *Mal'tsev term* and congruence-permutable varieties are called *Mal'tsev varieties*. Any variety that contains a group operation $\cdot$ is congruence-permutable, and the Mal'tsev term is $xy^{-1}z$.

Smith commutator

For an algebra X in a Mal'tsev variety $\mathcal{V}$ with Mal'tsev term $q(x, y, z)$ and two congruences α and β on X, the Smith commutator $[\alpha, \beta]_S$ is the smallest congruence θ on X for which the mapping

$$q \colon \{\, (x, y, z) \mid (x, y) \in \alpha \text{ and } (y, z) \in \beta \,\} \to X/\theta$$

that sends (x, y, z) to the θ-class of $q(x, y, z)$ is a morphism.

Commutators for skew braces: Huq = Smith

In the category of left skew braces, the Huq commutator is equal to the Smith commutator.

Commutators for skew braces: Huq = Smith

Proposition

If I and J are two ideals of a left skew brace $(A, *, \circ)$, their (Huq=Smith) commutator $[I, J]$ is the ideal of A generated by the union of the following three sets:
(1) *the set $\{\, i \circ j \circ (j \circ i)' \mid i \in I,\ j \in J \,\}$, (which generates the commutator $[I, J]_{(A,\circ)}$ of the normal subgroups I and J of the group $(A, \circ)$);*
(2) *the set $\{\, i * j * (j * i)^{-1} \mid i \in I,\ j \in J \,\}$, (which generates the commutator $[I, J]_{(A,*)}$ of the normal subgroups I and J of the group $(A, *)$); and*
(3) *the set $\{\, (i \circ j) * (i * j)^{-1} \mid i \in I,\ j \in J \,\}$.*

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Huq $\neq$ Smith for near-rings, loops, digroups,...

Now that we have a commutator for a left skew brace, we immediately have:

prime ideals in a left skew brace, semiprime ideals, the Zariski spectrum of any left skew brace, hyperabelian skew braces (those with no prime ideal), abelian left skew braces A (those for which $[A, A] = 1$), solvable, nilpotent skew braces, the Jacobson radical, centralizers, center,...

Prime and semiprime ideals in a left skew brace

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An ideal I of a left skew brace A is *prime* if for every pair of ideals J, K of A, $[J, K] \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$;

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An ideal I of a left skew brace A is *prime* if for every pair of ideals J, K of A, $[J, K] \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$; equivalently, if and only if I is semiprime and I is $\wedge$-irreducible, that is, the intersection of two ideals of A that properly contain I properly contains I.

Centralizer

We have that $I \vee J = I * J = I \circ J$, $[I, J] = [J, I]$, $[I * J, K] = [I, K] * [J, K]$ for all ideals I, J, K of a left skew brace A.

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Every ideal I has a *centralizer* $C_{(A,*,\circ)}(I)$: it is the greatest ideal of A contained in $C_{(A,*)}(I) \cap C_{(A,\circ)}(I) \cap \ker(\lambda|^I \colon (A, \circ) \to \mathrm{Aut}(I, *))$.

Semidirect products of left skew braces

Proposition

Let $(A, *, \circ)$ be a left skew brace, B a skew subbrace of A and I an ideal of A. The following conditions are equivalent:
(1) $A = B \circ I$ *and* $B \cap I = \{1_A\}$.
(2) *For every* $a \in A$, *there are unique* $b \in B$ *and* $i_1 \in I$ *such that* $a = b \circ i_1$.
(3) *For every* $a \in A$, *there are unique* $b \in B$ *and* $i_2 \in I$ *such that* $a = i_2 \circ b$.
(4) $A = B * I$ *and* $B \cap I = \{1_A\}$.
(5) *For every* $a \in A$, *there are unique* $b \in B$ *and* $i_3 \in I$ *such that* $a = b * i_3$.
(6) *For every* $a \in A$, *there are unique* $b \in B$ *and* $i_2 \in I$ *such that* $a = i_4 * b$.
(7) *There exists a left skew brace morphism* $A \to B$ *whose restriction to* B *is the identity and whose kernel is* I.
(8) *There is a left skew brace idempotent endomorphism of* A *whose image is* B *and whose kernel is* I.

Semidirect products of left skew braces

In this case, we say that A is a semidirect product of B and I: $A = I \rtimes B$.

Semidirect products of left skew braces

Let $(Y, *, \circ)$ and $(K, *, \circ)$ be a pair of digroups, $\phi_*: (G, *) \to \mathrm{Aut}(K, *)$ a $(G, *)$-action on $(K, *)$ and $\phi_\circ: (G, \circ) \to \mathrm{Aut}(K, \circ)$ a $(G, \circ)$-action on $(K, \circ)$. Then the set $Y \times K$ is endowed with a digroup structure with the two associated semi-direct product laws, which we shall again respectively denote by $*$ and $\circ$.

Semidirect products of left skew braces

Theorem

Let $(Y,*,\circ)$ and $(K,*,\circ)$ be a pair of left skew braces. The digroup $(K \rtimes Y,*,\circ)$ is a left skew brace if and only if
(a) *the action $\phi_\circ$ of $(Y,\circ)$ on $(K,\circ)$ is a group homomorphism* $\phi_\circ\colon (Y,\circ) \to \mathsf{Aut}_{\mathsf{SKB}}(K,*,\circ)$.
(b) *the action ϕ_* of $(Y,*)$ on $(K,*)$ is a group homomorphism* $\phi_*\colon (Y,\circ) \to \mathsf{Aut}_{(K,\circ)\text{-}\mathsf{Gp}}((K,*)$, *the automorphism group of $(K,*)$ as an object in the category $K,\circ)$-Gp of $(K,\circ)$-groups.*
(c) $\phi_{*(y\circ y')} = \phi_{\circ y}\phi_{*y'}\phi_{\circ y}^{-1}\phi_{*y}$ *for all $(y,y') \in Y \times Y$.*

Semidirect products of left skew braces

Corollary

Let $(Y,*,\circ)$ and $(K,*,\circ)$ be a pair of digroups, ϕ_* a $(G,*)$-action on $(K,*)$ and $\phi_\circ$ a $(G,\circ)$-action on $(K,\circ)$. Then the digroup $(K \rtimes Y,*,\circ)$ is a left skew brace if and only if the mapping

$$\Phi\colon\ Y \rtimes_\lambda Y \to \mathrm{Aut}_{(K,\circ)\text{-}\mathrm{Gp}}((K,*)^{\mathrm{op}} \rtimes \mathrm{Aut}_{\mathrm{SKB}}((K,*,\circ)$$
$$\Phi\colon (y,y') \mapsto (\phi_{*y},\phi_{\circ y'}),$$

is a group morphism.