On the enumeration of finite L-algebras

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Abstract

We use Constraint Satisfaction Methods to construct and enumerate finite L-algebras up to isomorphism. These objects were recently introduced by Rump and appear in Garside theory, algebraic logic, and the study of the combinatorial Yang–Baxter equation. There are 377322225 isomorphism classes of L-algebras of size eight. [...]

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Introduction

By Rump, a unital cycloid is a set X with a distinguished element $e \in X$ - the logical unit - and a binary operation $\to$ on X such that the following axioms are fulfilled for any $x, y, z \in X$:

$$x \to e = x \to x = e \tag{L1}$$
$$e \to x = x \tag{L2}$$
$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z). \tag{L3}$$

If furthermore the condition

$$x \to y = y \to x = e \Leftrightarrow x = y \tag{L4}$$

is fulfilled, we speak of an L-algebra.
Equation (L3) is also known as the cycloid equation, an equation present in the algebraic theory of the Yang-Baxter equation, Garside theory, algebraic logics and even number theory.
Due to the growing interest in L-algebras, it is important to have small examples to work with. For cycle sets, this has recently been done by Akgün, Mereb, Vendramin, who constructed a database of nondegenerate cycle sets of size ≤ 10 using Constraint Satisfaction methods. In our work, we apply these methods in order to construct a database of all non-isomorphic L-algebras of size ≤ 8.

L-algebras and order

An L-algebra has a natural order given by

$$x \leq y :\Leftrightarrow \; x \to y = e.$$

If X is a logical algebra - such as the Lindenbaum-Tarski-algebra of a theory - then this can be interpreted as x being logically stronger than y.
With respect to the natural order, the left-multiplications in an L-algebra are monotonous, meaning that for $x, y, z \in X$, we have the implication

$$y \leq z \Rightarrow \; x \to y \leq x \to z.$$

A $\wedge$-semilattice is a set X with a binary operation $\wedge$ on X that is *idempotent*, *associative* and *commutative*. Each $\wedge$-semilattice X comes with a natural order given by $x \leq y :\Leftrightarrow x \wedge y = x$. Under this partial order, $x \wedge y$ is the greatest lower bound for x and y in X.

A $\wedge$-semibrace is a set X with a logical unit $e \in X$ and two binary operations $\wedge, \to$ on X such that $(X, \wedge)$ is a $\wedge$-semilattice and, furthermore, the following axioms are fulfilled for any $x, y, z \in X$:

$$x \to e = x \to x = e \tag{S1}$$
$$e \to x = x \tag{S2}$$
$$x \to (y \wedge z) = (x \to y) \wedge (x \to z) \tag{S3}$$
$$(x \wedge y) \to z = (x \to y) \to (x \to z). \tag{S4}$$

Note that all these axioms can be interpreted as statements in propositional logic!
By the commutativity of $\wedge$, equation (S4) implies that the triple $(X, \to, e)$ is in fact an L-algebra!

By the following theorem of Rump, each L-algebra gives rise to a $\wedge$-semibrace:

Theorem (Rump): *Let X be an L-algebra. Then there is a $\wedge$-semibrace $C(X)$ - the $\wedge$-closure - such that $X \subseteq C(X)$ is a sub-L-algebra and each element $y \in C(X)$ can be represented as $y = \bigwedge_{i=1}^{n} x_i$ with all $x_i \in X$.*
The $\wedge$-closure of an L-algebra is unique up to isomorphism of $\wedge$-semibraces.

Viewing L-algebras as logical semantics, Rump's theorem says that L-algebras are fragments of logical semantics with operations $\wedge, \to$ that fulfill the semibrace axioms. In Garside theory, the theorem reflects the construction of a Garside interval from a Garside germ.

Constraint Satisfaction Methods

Constraint satisfaction methods have been successfully applied by Akgün, Mereb and Vendramin for computing the non-isomorphic cycle sets of size ≤ 10.
We use the constraint modeling assistant Savile Row, which is capable of constructing solutions to combinatorial problems. In order to compute non-isomorphic L-algebras of size n, we encode an L-algebra as a $n \times n$-matrix $M := (M[i,j])$ with entries in $\{1, \dots, n\}$ where $M[i,j] = i \to j$ and set n as the logical unit of the L-algebra. The L-algebra axioms then translate to the following constraints:

$$\forall 1 \leq i \leq n : M[i,n] = n \wedge M[i,i] = n \tag{L1'}$$
$$\forall 1 \leq i \leq n : M[n,i] = i \tag{L2'}$$
$$\forall 1 \leq i,j,k \leq n : M[M[i,j], M[i,k]] = M[M[j,i], M[j,k]] \tag{L3'}$$
$$\forall 1 \leq i < j \leq n : M[i,j] + M[j,i] < 2n. \tag{L4'}$$

Note that the last condition ensures that $M[i,j]$ and $M[j,i]$ are never equal to n when $i \neq j$.
In order to construct *non-isomorphic* L-algebras we lexicographical symmetry breaking, meaning that M has to be the lexicographically least representant in its conjugacy class. Here two $n \times n$-matrices M and N with entries in $\{1, \dots, n\}$ are said to be conjugate if there is a permutation $\pi \in \mathrm{Sym}_{n-1}$ such that $N[i,j] = \pi^{-1}(M[\pi(i), \pi(j)])$. This can be translated to a constraint that avoids isomorphism checking for solutions.
Using these methods, we can effectively compute all L-algebras of size ≤ 7.

Splitting the search case

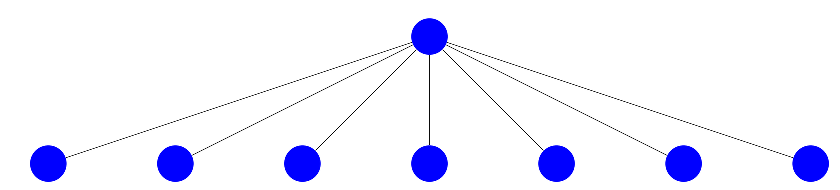
In order to calculate L-algebras of size 8, it is necessary to split the search space. An efficient way to achieve that is to search for L-algebras with a given canonical order.
Therefore, we calculate all non-isomorphic partially ordered sets on $\{1, \dots, n-1\}$ in order to represent the order on the non-unit elements of an L-algebra of size n. Given a partial order $\leq_P$ on $\{1, \dots, n-1\}$, we translate the following condition into a constraint:

$$\forall 1 \leq i,j < n : i \leq_P j \Leftrightarrow M[i,j] = n$$

We furthermore use a variant of the lexicographical symmetry breaking that ensures that the matrix M is the lexicographically least matrix with respect to symmetries preserving the order relation $\leq_P$.

In order to accelerate the search process, we use equations that follow from the partial order relation and the L-algebra axioms.
For $n = 8$, there are 2045 partially ordered sets to consider. In all but two cases (see below), this approach leads to a relatively fast computation of the involved L-algebras.
In total, we get 89712587 non-isomorphic L-algebras that do not fall under the discrete or diamond case.

The discrete case



In the case that the L-algebra is discrete, meaning that the order on the non-unit elements is discrete (see above), we use a different approach. This case is computationally hard, mainly because the underlying partially ordered set has $\mathrm{Aut}(P) \cong \mathrm{Sym}_{n-1}$ as its automorphism group and the order does not give any nontrivial restrictions on the entries $M[i,j] = i \to j$.
In this case we work with the $\wedge$-closure $C(X)$. Denoting $X' := X \setminus \{e\}$, we model a $\wedge$-semilattice generated by X as a semilattice closure operator

$$\mathrm{cl} : \mathcal{P}(X') \to \mathcal{P}(X')$$
$$A \mapsto \overline{A}$$

subject to the following relations:

$$\overline{\emptyset} = \emptyset \tag{C1}$$
$$A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B} \quad \text{for } A, B \in \mathcal{P}(X') \tag{C2}$$
$$A \subseteq \overline{B} \Rightarrow \overline{A} \subseteq \overline{B} \quad \text{for } A, B \in \mathcal{P}(X') \tag{C3}$$
$$x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}} \quad \text{for } x, y \in X'. \tag{C4}$$

In $C(X)$, these closure operators describe the closure operators

$$\overline{A} = \{x \in X' : x \geq \bigwedge A\}.$$

As there are too many operators of this form to be computationally tractable, we look at the restriction of a semilattice closure operator to arguments in $\binom{X'}{\leq k}$, the subsets of size k and less. Such an operator is called a k-approximate semilattice.
Furthermore, by a theorem of Rump, $C(X)$ is a *lower semimodular semilattice* which translates to the following restricted lower semimodularity condition for the operator $\mathrm{cl}(A) = \overline{A}$:

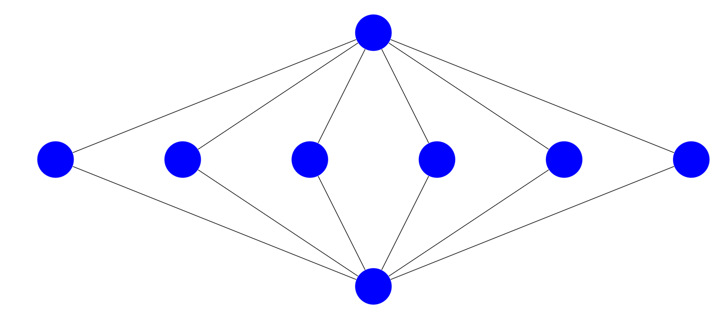
$$\overline{A} \subseteq \overline{B} \subseteq \overline{A \cup \{x\}} \Rightarrow (\overline{A} = \overline{B} \text{ or } \overline{B} = \overline{A \cup \{x\}})$$
$$\text{for } A, B \in \mathcal{P}(X');\; x \neq e.$$

Using Savile Row, we compute 73 3-approximate semilattices on $\{1, \dots, 7\}$, fulfilling restricted lower semimodularity. For such an operator $A \mapsto \overline{A}$, the following conditions, amongst others, allow an efficient computation of the associated discrete L-algebras:

$$z \in \overline{\{x,y\}} \Leftrightarrow (x \to y) \to (x \to z) = e$$
$$\Leftrightarrow (x \wedge y) \to z = e$$
$$z \in \overline{\{w,x,y\}} \Leftrightarrow \quad ((w \to x) \to (w \to y))$$
$$\to ((w \to x) \to (w \to z)) = e$$
$$\Leftrightarrow (w \wedge x \wedge y) \to z = e.$$

Similar to the case of partially ordered sets, we use lexicographical symmetry breaking with respect to symmetries of the given 3-approximate semilattices.
In total, we get 149390095 non-isomorphic discrete L-algebras of size 8.

The diamond case



The case of a diamond order (see above) is the second hardest, mainly due to a high number of symmetries and very few restrictions on the values of $M[i,j] = i \to j$. Also, there is no known restriction on the semilattice structure of $C(X)$, whenever X is a non-discrete L-algebra, that leads to a reasonable splitting of the search space.
We tackle this case as follows: we first construct all non-isomorphic graphs on the set $\{1, \dots, n-2\}$. Given such a graph Γ, described as a collection of two-element sets $\{i,j\} \subseteq \{1, \dots, n-2\}$, we demand that

$$\{i,j\} \in \Gamma$$
$$\Leftrightarrow \forall 1 \leq k \leq n-2 : M[M[i,j], M[i,k]] = n$$
$$(\Leftrightarrow \forall 1 \leq k \leq n-2 : i \wedge j \leq k \text{ in } C(X).)$$

Again, we search for the lexicographically least matrix M with respect to the symmetries of Γ.
By parallelizing the search over 156 graphs, we compute 138219543 non-isomorphic L-algebras on a diamond poset of size 8.

L-algebras of size ≤ 8

Denoting by $L(n)$ the number of non-isomorphic L-algebras of size n, we get the following results:

Theorem (Dietzel, Menchón, Vendramin): *For $n \leq 8$, the values of $L(n)$ are given by the following table:*

n	1	2	3	4	5	6	7	8
$L(n)$	1	1	5	44	632	15582	907806	377322225

The following table summarizes the results of our research, together with some computational data.

	discrete	non-discrete	
Poset	trivial	diamond	other
Nr. cases	73	156	2043
Nr. solutions	149390095	138219543	89712587
Run-time	4 days	5 hours	11 hours
Data (compressed)	208MB	212MB	153MB

Reference

[1] C. Dietzel, P. Menchón, and L. Vendramin.
On the enumeration of finite L-algebras.
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