New set-theoretic solutions of the Yang-Baxter equation through weak braces

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“The algebra of the Yang-Baxter equation"

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Outline

1. An overview on some algebraic structures connected to the Yang-Baxter equation

2. The algebraic structures of weak braces

3. Quotient structures of special weak braces

4. Weak braces through Rota-Baxter operators

An overview on some algebraic structures connected to the Yang-Baxter equation

Solutions of the Yang-Baxter equation

Definition (Drinfel'd, 1992)

If S is a set, a map $r : S \times S \longrightarrow S \times S$ satisfying the braid relation

$$(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)$$

will be called *set-theoretic solution*, or briefly *solution*, of the Yang-Baxter equation.

For a solution r, we introduce two maps λ_a and ρ_b from S into itself, and write

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in S$. In particular, the solution r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in S$;
- *right non-degenerate* if ρ_b is bijective, for every $b \in S$;
- *non-degenerate* if r is both left and right non-degenerate;
- *degenerate* if r is neither left nor right non-degenerate.

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Solutions associated to skew braces

[Rump, 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*. Interesting generalizations have been produced over the years.

[Guarnieri, Vendramin, 2017]

A triple $(S,+,\circ)$ is called *skew brace* if $(S,+)$ and $(S,\circ)$ are groups and it holds

$$\forall a,b,c \in S \qquad a\circ(b+c) = a\circ b - a + a\circ c.$$

If $(S,+)$ is abelian, then $(S,+,\circ)$ is a *brace*.

Every skew brace $(S,+,\circ)$ gives rise to a *non-degenerate bijective solution*

$$r_S(a,b) = \left(\lambda_a(b), \lambda^{-1}_{\lambda_a(b)}(-a\circ b + a + a\circ b)\right),$$

with $\lambda_a(b) = -a + a\circ b$. Moreover, r_S is *involutive*, i.e., $r^2 = \mathrm{id}_{S\times S}$, if and only if $(S,+,\circ)$ is a brace.

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The opposite skew brace

If $(S,+,\circ)$ is a skew brace, one can consider the structure

$$S^{op} = (S, +^{op}, \circ)$$

with $a +^{op} b = b + a$, for all $a, b \in S$. Then, S^{op} is a skew brace, called the *opposite skew brace* of $(S,+,\circ)$.

As shown by **[Koch, Truman, 2020]**, the solution $r_{S^{op}}$ associated to S^{op} is nothing but the inverse of the solution

$$r_S(a,b) = \left(\lambda_a(b)\,, \lambda^{-1}_{\lambda_a(b)}(-a \circ b + a + a \circ b)\right),$$

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$$r_S^{-1} = r_{S^{op}}.$$

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Solutions associated to semi-braces

[Catino, Colazzo, Stefanelli, 2017]

A *(left cancellative) semi-brace* is a triple $(S, +, \circ)$ such that $(S, +)$ is a left cancellative semigroup, $(S, \circ)$ is a group, and
$\forall a, b, c \in S \qquad a \circ (b + c) = a \circ b + a \circ (a^- + c)$.

Every skew brace is a semi-brace. If $(S, +, \circ)$ is a (left cancellative) semi-brace, the map

$$r_S(a, b) = \left(a \circ (a^- + b), (a^- + b)^- \circ b\right)$$

is a *left non-degenerate (not necessarily bijective) solution*, with $\lambda_a(b) = a \circ (a^- + b)$.

Later, **[Jespers, Van Antwerpen, 2019]** generalized semi-braces assuming that $(S, +)$ is a ~~(left cancellative)~~ semigroup.
In this case, if ρ_b is an anti-homomorphism of S, then r_S is a *degenerate solution*.

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Inverse semigroups

What happens if $(S, \circ)$ ~~group~~ $\longrightarrow (S, \circ)$ inverse semigroup?

Definition

A semigroup S is called *inverse* if, for each $a \in S$, there exists a unique $a^{-1} \in S$ satisfying

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$

Every group is an inverse semigroup and they hold:

- $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$, for all $a, b \in S$;

- $E(S)$ is an inverse subsemigroup of S;

- $\forall a \in S \quad aa^{-1}, a^{-1}a \in E(S)$ and all the idempotents are of this form.

Inverse semigroup theory was initiated in the 1950s and it has been extensively studied during the years. Most of the known results up to the early 1980s are summarized into the monograph by **[Petrich, 1984]**.

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Inverse semi-braces and solutions associated

[Catino, Stefanelli, M., 2021]

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In this case, we provide sufficient conditions for the map

$$r_S(a,b)=\left(a\circ(a^{-}+b),(a^{-}+b)^{-}\circ b\right)$$

to be a solution. Such solutions are *degenerate and not bijective*.

Example

If $(S,\circ)$ is an inverse semigroup and $(S,+)$ is a right zero semigroup, then $(S,+,\circ)$ is an inverse semi-brace for which the map

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Special inverse semi-braces

Inverse semigroups in which $aa^{-1} = a^{-1}a$, for every $a \in S$, are called *Clifford semigroups*. Any Clifford semigroup $(S, \circ)$ gives rise to two inverse semi-braces

$$S = (S, +, \circ) \quad a + b = a \circ b \qquad\qquad \tilde{S} = (S, \tilde{+}, \circ) \quad a \,\tilde{+}\, b = b \circ a,$$

in which also $(S, +)$ is a Clifford semigroup.
They give rise to two solutions

$$r_S(a,b) = \left(a \circ a^- \circ b,\ b^- \circ a \circ b\right) \qquad r_{\tilde{S}}(a,b) = \left(a \circ b \circ a^-,\ a \circ b^- \circ b\right),$$

for all $a, b \in S$, respectively.

Work program

Studying inverse semi-braces having $(S, +)$ as an inverse semigroup.

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$$r_S(a, b) = \left(a \circ a^- \circ b,\ b^- \circ a \circ b\right) \qquad r_{\tilde{S}}(a, b) = \left(a \circ b \circ a^-,\ a \circ b^- \circ b\right),$$

for all $a, b \in S$, respectively.

Work program

Studying inverse semi-braces having $(S, +)$ as an inverse semigroup.

The algebraic structures of weak braces

Weak braces

Definition (Catino, Miccoli, Stefanelli, M. (2022))

A triple $(S,+,\circ)$ is called *weak brace* if
- $(S,+)$ and $(S,\circ)$ are inverse semigroups;
- $\forall a,b,c\in S \quad a\circ(b+c)=a\circ b-a+a\circ c$,
- $\forall a\in S \qquad a\circ a^{-}=-a+a$,

where $-a$ and a^{-} denote the inverses of $(S,+)$ and $(S,\circ)$.

- Any skew brace is a weak brace since $(S,+)$ and $(S,\circ)$ are groups and

$$a\circ a^{-}=-a+a=0.$$

- Any weak brace is an inverse semi-brace since

$$\forall a,b\in S \qquad \lambda_a(b)=-a+a\circ b=a\circ\left(a^{-}+b\right).$$

- Given a weak brace $(S,+,\circ)$, we can consider its *opposite weak brace* $S^{op}=(S,+^{op},\circ)$, with $a+^{op}b=b+a$, for all $a,b\in S$.

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Any weak brace gives rise to a solution

Theorem

If $(S,+,\circ)$ is a weak brace, then the map

$$r_S(a,b)=\left(a\circ\left(a^-+b\right),\left(a^-+b\right)^-\circ b\right),$$

for all $a,b\in S$, is a *degenerate solution*. Moreover, r_S has a *behaviour near to bijectivity*.

Indeed, the solution $r_{S^{op}}$ associated to the opposite weak brace S^{op} of S is such that

$$r_S\,r_{S^{op}}\,r_s=r_S,\qquad r_{S^{op}}\,r_S\,r_{S^{op}}=r_{S^{op}},\qquad \text{and}\qquad rr_{S^{op}}=r_{S^{op}}r.$$

We recall that in the case of S a skew brace, $r_{S^{op}}=r_S^{-1}$.

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Weak braces as a special subclass of inverse semi-braces

Work program

Studying inverse semi-braces having $(S,+)$ as an inverse semigroup.

Proposition

A triple $(S,+,\circ)$ is a *weak brace* if and only if $(S,+,\circ)$ is an *inverse semi-brace* such that:
- $(S,+)$ is an inverse semigroup,
- $\lambda_a(b) = a\circ(a^- + b) = -a + a\circ b$,

for all $a,b\in S$.

What about the structure of any weak brace $(S,+,\circ)$?
- $(S,+)$ is a **Clifford semigroup**;
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Quotient structures of special weak braces

Dual weak braces

If $(S,+,\circ)$ is a weak brace, in general, $(S,\circ)$ is not a Clifford semigroup.

Definition (Catino, Stefanelli, M., preprint (2022))

We call *dual weak brace* any weak brace having $(S,\circ)$ as a Clifford semigroup.

The solution associated to any dual weak brace has a *behaviour near to non-degeneracity*, since

$$\lambda_a\lambda_{a^-}\lambda_a = \lambda_a, \qquad \lambda_{a^-}\lambda_a\lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \quad \lambda_a\lambda_{a^-} = \lambda_{a^-}\lambda_a,$$
$$\rho_b\rho_{b^-}\rho_b = \rho_b, \qquad \rho_{b^-}\rho_b\rho_{b^-} = \rho_{b^-}, \quad \text{and} \quad \rho_b\rho_{b^-} = \rho_{b^-}\rho_b,$$

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For this special weak brace, we can consider the notion of *ideal* and, consequently, that of quotient structure.

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Normal subsets of a Clifford semigroup

For inverse semigroups, it is possible to mimic the group-theoretical treatment of congruences since one can find special subsemigroups that are analogue of normal subgroups of groups.

Definition (Petrich, Reilly, 1999)

Let S be a Clifford semigroup. A subset N of S is *normal* if it satisfies the following conditions:
- $E(S) \subseteq N$,
- $\forall\, a \in S \quad a \in N \Longrightarrow a^{-1} \in N$,
- $\forall\, a, b \in S \quad a,\ a^{-1}ab \in N \Longrightarrow ab \in N$,
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Clearly, $E(S)$ is a normal subset of S.

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Ideals of dual weak braces

Definition (Catino, Stefanelli, M., preprint (2022))

Let $(S, +, \circ)$ be a dual weak brace. Then, a subset N of S is an *ideal* of S if the following hold:

- N is a normal subset of $(S, +)$ and $(S, \circ)$,
- $\lambda_a(N) \subseteq N$, for every $a \in S$.

Examples

- S and $E(S)$ are trivial ideals of S.
- The set

$$\mathrm{Soc}(S) = \{a \mid a \in S,\ \forall\, b \in S \quad a + b = a \circ b \quad \text{and} \quad a + b = b + a\}$$

is an ideal of $(S, +, \circ)$ called *socle* of S.

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Quotient structures of dual weak braces

We can consider the quotient structure of any dual weak brace by means of its ideals.

Theorem (Catino, Stefanelli, M., preprint (2022))

Let N be an ideal of a dual weak brace $(S,+,\circ)$. Then, the relation $\sim_N$ on S given by

$$\forall\, a,b\in S \quad a\sim_N b \iff a-a=b-b \text{ and } -a+b\in N,$$

is a congruence of $(S,+,\circ)$.
Moreover, $S/\sim_N$ is a dual weak brace with semilattice of idempotents isomorphic to $E(S)$.

Weak braces through Rota-Baxter operators

Rota-Baxter operators on Clifford semigroups

[Guo, Lang, Sheng, 2020] firstly introduced the notion of Rota–Baxter operator on groups, pursued further by **[Bardakov, Gubarev, 2021]**.

Definition (Catino, Stefanelli, M., preprint (2022))

If $(S,+)$ is a Clifford semigroup, any map $\mathfrak{R}: S \to S$ satisfying

$$\mathfrak{R}(a)+\mathfrak{R}(b)=\mathfrak{R}(a+\mathfrak{R}(a)+b-\mathfrak{R}(a))$$
$$a+\mathfrak{R}(a)-\mathfrak{R}(a)=a,$$

for all $a,b \in S$, is called *Rota–Baxter operator* on $(S,+)$.

Any endomorphism of a Clifford semigroup $(S,+)$ that is a Rota–Baxter operator is called *Rota–Baxter endomorphism* of $(S,+)$.

For instance, if φ is an idempotent endomorphism such that $\varphi(e)=e$, for every $e \in E(S,+)$, then the map $\mathfrak{R} := -\varphi$ is such an example.

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A description of all idempotent RB-endomorphisms of groups

Theorem (Catino, Stefanelli, M., preprint (2022))

Let $(G,+)$ be a group and consider:

- $N \trianglelefteq G$ such that G/N is abelian,
- $\mathscr{S}$ a set of representatives of G/N that is a subgroup of G.

Then, any map $\mathfrak{R}: G \to G$ such that

$$\operatorname{Im}\mathfrak{R} = \mathscr{S} \quad \text{and} \quad \mathfrak{R}(g) \in N + g,$$

for every $g \in G$, is an idempotent RB-endomorphism of G.

Conversely, if $\mathfrak{R}$ is an idempotent RB-endomorphism of G, then

- $G/\ker\mathfrak{R}$ is abelian,
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Dual weak braces through Rota–Baxter operators

Rota-Baxter operators on groups give rise to new skew braces as proved by **[Bardakov, Gubarev, 2021]**. In the same way,

Proposition (Catino, Stefanelli, M., preprint (2022))

If $\mathfrak{R} : S \to S$ is a Rota–Baxter operator on a Clifford semigroup $(S,+)$, set

$$\forall a,b \in S \quad a \circ b := a + \mathfrak{R}(a) + b - \mathfrak{R}(a),$$

then $(S,+,\circ)$ is a dual weak brace which we denote by $S_{\mathfrak{R}}$.

For instance, considered the previous class of examples $\mathfrak{R} := -\varphi$,

- if $\varphi(a) = a - a$, for every $a \in S$, then $a \circ b = a + b$,
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Ideals of the dual weak braces $S_{\mathfrak{R}}$

The following result describes the ideals of dual weak braces associated to Rota–Baxter operators.
Into the specific, the request of *λ-invariance is redundant*.

Proposition (Catino, Stefanelli, M., preprint (2022))

Let $(S, +)$ be a Clifford semigroup.
Consider the dual weak brace $S_{\mathfrak{R}}$ obtained by a Rota–Baxter operator $\mathfrak{R}$ on $(S, +)$ and $N \subseteq S$. Then,

$$\text{N is an ideal of } S_{\mathfrak{R}} \iff \text{N both is a normal subset of the Clifford semigroups } (S, +) \text{ and } (S, \circ).$$

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$$\text{N is an ideal of } S_{\mathfrak{R}} \iff \text{N both is a normal subset of the Clifford semigroups } (S,+) \text{ and } (S,\circ).$$

Some references

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