

Some new results on bi-skew braces

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Definition ([Guarnieri and Vendramin, 2017])

A *skew brace* is a triple (A, \cdot, \circ) , where (A, \cdot) , (A, \circ) are groups and

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c).$$

Here a^{-1} denotes the inverse of a in (A, \cdot) .

Skew braces are connected with

- radical rings;
- solutions of the set-theoretic Yang–Baxter equation;
- regular subgroups of holomorphs of groups;
- Hopf–Galois structures.

Definition ([Childs, 2019])

A *bi-skew brace* is a skew brace (A, \cdot, \circ) such that also (A, \circ, \cdot) is a skew brace.

Example

Let (A, \circ) be a group.

- A *trivial* skew brace (A, \circ, \circ) is a bi-skew brace.
- An *almost trivial* skew brace $(A, \circ_{\text{op}}, \circ)$, where $a \circ_{\text{op}} b = b \circ a$, is a bi-skew brace.

Example

$(\mathbb{Z}, +, \tilde{\circ})$ is a bi-skew brace, where $a \tilde{\circ} b = a + (-1)^a b$.

Byott's conjecture and bi-skew braces

Conjecture (Byott's conjecture)

Let (A, \cdot, \circ) be a finite skew brace. If (A, \cdot) is soluble, then (A, \circ) is soluble.

Theorem ([LS and Trappeniers, 2022])

Let (A, \cdot, \circ) be a bi-skew brace. Then (A, \cdot) is soluble if and only if (A, \circ) is soluble.

A key player

Recall that *ideals* of skew braces are the substructures to consider in order to define quotient skew braces.

Fact

Let (A, \cdot, \circ) be a bi-skew brace. Then there exists an ideal I of (A, \cdot, \circ) such that

- (I, \cdot, \circ) is a trivial skew brace;
- $(A/I, \cdot, \circ)$ is an almost trivial skew brace.

(Here $I = A_{\text{op}}^2$.)

The proof and a generalisation

Proof.

Suppose that (A, \cdot) is soluble. Then (I, \cdot) and $(A/I, \cdot)$ are soluble. As $(I, \cdot) = (I, \circ)$ and $(A/I, \cdot) \cong (A/I, \circ)$, also (I, \circ) and $(A/I, \circ)$ are soluble. We conclude that (A, \circ) is soluble. \square

Fact

This argument works also for skew braces (A, \cdot, \circ) admitting series $1 = A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_2 \subseteq A_1 = A$ where A_{i+1} is an ideal of A_i with $(A_i/A_{i+1}, \cdot) \cong (A_i/A_{i+1}, \circ)$ for all i .

In particular, Byott's conjecture holds for soluble skew braces, so also for skew braces that are right nilpotent, left nilpotent, strongly, central nilpotent, metatrivial, γ -homomorphic, ...

A classification problem

Recall that $(\mathbb{Z}, +, \tilde{\circ})$ is a bi-skew brace, where $a \tilde{\circ} b = a + (-1)^a b$.

Proposition ([Rump, 2007])

Let $(\mathbb{Z}, +, \circ)$ be a skew brace. If $(\mathbb{Z}, +, \circ)$ is not trivial, then $(\mathbb{Z}, \circ) = (\mathbb{Z}, \tilde{\circ})$.

Problem ([Vendramin, 2019])

Determine the skew braces of the form $(\mathbb{Z}, \cdot, +)$.

Theorem ([Cedó et al., 2019])

Let $(\mathbb{Z}, \cdot, +)$ be a skew brace such that (\mathbb{Z}, \cdot) is abelian. Then $(\mathbb{Z}, \cdot, +)$ is trivial.

The resolution

We remark that we always have the skew braces $(\mathbb{Z}, +, +)$, $(\mathbb{Z}, \tilde{\circ}, +)$, and $(\mathbb{Z}, \tilde{\circ}_{\text{op}}, +)$.

Theorem ([LS and Trappeniers, 2022])

Let $(\mathbb{Z}, \cdot, +)$ be a skew brace. If $(\mathbb{Z}, \cdot, +)$ is not trivial, then $(\mathbb{Z}, \cdot) = (\mathbb{Z}, \tilde{\circ})$ or $(\mathbb{Z}, \cdot) = (\mathbb{Z}, \tilde{\circ}_{\text{op}})$.

Sketch of the proof.

- If (\mathbb{Z}, \cdot) is abelian, then $(\mathbb{Z}, \cdot, +)$ is trivial.
- If (\mathbb{Z}, \cdot) is not abelian, then we can show that either $(\mathbb{Z}, \cdot, +)$ or $(\mathbb{Z}, \cdot_{\text{op}}, +)$ is a bi-skew brace.

In particular, either $(\mathbb{Z}, +, \cdot)$ or $(\mathbb{Z}, +, \cdot_{\text{op}})$ is a skew brace.

We obtain that either $(\mathbb{Z}, \cdot) = (\mathbb{Z}, \tilde{\circ})$ or $(\mathbb{Z}, \cdot_{\text{op}}) = (\mathbb{Z}, \tilde{\circ})$. \square

The Yang–Baxter equation

Let (X, r) be a (*nondegenerate bijective*) *solution* (of the *set-theoretic Yang–Baxter equation*). Write

- $r(x, y) = (\sigma_x(y), \tau_y(x))$;
- $r^{-1}(x, y) = (\hat{\sigma}_x(y), \hat{\tau}_y(x))$.

Define the *structure group* of (X, r) to be

$$G(X, r) = \langle X \mid x \circ y = \sigma_x(y) \circ \tau_y(x) \rangle;$$

then (X, r) is *injective* if the natural map $X \rightarrow G(X, r)$ is injective.

For example, this is the case when (X, r) is *involutive*, that is, when $r^2 = \text{id}$.

We remark that $G(X, r)$ has a natural structure of a skew brace.

Bi-skew braces and the Yang–Baxter equation

Question

When is $G(X, r)$ a bi-skew brace?

Theorem ([LS and Trappeniers, 2022])

Let (X, r) be a solution.

- If for all $x, y \in X$,

$$\sigma_{\hat{\sigma}_x(y)} = \sigma_y,$$

then $G(X, r)$ is a bi-skew brace.

- If (X, r) is injective, then also the opposite implication holds.

Fact

When (X, r) is involutive, we find solutions studied in [Jedlička et al., 2020].

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