Two-sided skew braces

Senne Trappeniers

July 11, 2022

VRIJE
UNIVERSITEIT
BRUSSEL

Jacobson radical rings

Definition

A ring R is a *Jacobson radical ring* if it coincides with it's own Jacobson radical.

Let R be a ring, then $(R, \circ)$ with $a \circ b = a + b + ab$ is a monoid.

Jacobson radical rings

Definition

A ring R is a *Jacobson radical ring* if it coincides with it's own Jacobson radical.

Let R be a ring, then $(R, \circ)$ with $a \circ b = a + b + ab$ is a monoid.

Proposition

A ring R is a Jacobson radical ring if and only if $(R, \circ)$ is a group.

Jacobson radical rings

Definition

A ring R is a *Jacobson radical ring* if it coincides with it's own Jacobson radical.

Let R be a ring, then $(R, \circ)$ with $a \circ b = a + b + ab$ is a monoid.

Proposition

A ring R is a Jacobson radical ring if and only if $(R, \circ)$ is a group.

In this case, for all $a, b, c \in R$:

$$a \circ (b + c) = a \circ b - a + a \circ c, \tag{1}$$

$$(a + b) \circ c = a \circ c - c + b \circ c. \tag{2}$$

Braces

Observation Rump: Jacobson radical rings give set-theoretic solutions of the YBE.

Definition ([Rum07])

A *(left) brace* is a triple $(A, +, \circ)$ such that

1. $(A, +)$ is an abelian group,
2. $(A, \circ)$ is a group,
3. equation (1) holds.

If also equation (2) holds, then $(A, +, \circ)$ is a *two-sided brace*.

Braces

Observation Rump: Jacobson radical rings give set-theoretic solutions of the YBE.

Definition ([Rum07])

A *(left) brace* is a triple $(A, +, \circ)$ such that

1. $(A, +)$ is an abelian group,
2. $(A, \circ)$ is a group,
3. equation (1) holds.

If also equation (2) holds, then $(A, +, \circ)$ is a *two-sided brace*.

Bijective correspondence:

$$\begin{aligned} \{\text{two-sided braces}\} &\longleftrightarrow \{\text{Jacobson radical rings}\}, \\ (A, +, \circ) &\mapsto (A, +, *), \end{aligned}$$

where $a * b = -a + a \circ b - b$.

Skew braces

Definition ([GV17])

A *skew (left) brace* is a triple $(A, +, \circ)$ such that

1. $(A, +)$ is a group,
2. $(A, \circ)$ is a group,
3. equation (1) holds.

If also equation (2) holds, then $(A, +, \circ)$ is a *two-sided skew brace*.

Skew braces

Definition ([GV17])

A *skew (left) brace* is a triple $(A, +, \circ)$ such that

1. $(A, +)$ is a group,
2. $(A, \circ)$ is a group,
3. equation (1) holds.

If also equation (2) holds, then $(A, +, \circ)$ is a *two-sided skew brace*.

- Two-sided braces: well-studied,
- Two-sided skew braces: besides [Nas19], not much is known.

Opposite skew braces

Example (Trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ, \circ)$ is a two-sided skew brace.

Opposite skew braces

Example (Trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ, \circ)$ is a two-sided skew brace.

Example (Almost trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ_{\mathrm{op}}, \circ)$ is a two-sided skew brace.
Here $a \circ_{\mathrm{op}} b = b \circ a$.

Opposite skew braces

Example (Trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ, \circ)$ is a two-sided skew brace.

Example (Almost trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ_{\mathrm{op}}, \circ)$ is a two-sided skew brace. Here $a \circ_{\mathrm{op}} b = b \circ a$.

In general [KT20]: If $A = (A, +, \circ)$ is a skew brace, then $A_{\mathrm{op}} := (A, +_{\mathrm{op}}, \circ)$ is a skew brace, where $a +_{\mathrm{op}} b = b + a$. We call this the *opposite skew brace* of A

Ideals

Ideals of a skew brace $(A, \cdot, \circ)$ are precisely the sub skew braces I such that $(A/I, \cdot, \circ)$ is well-defined. In particular, I is a normal subgroup of $(A, +)$ and $(A, \circ)$.

$*$ revisited

Let A be a skew brace, then $a * b := -a + a \circ b - b$ gives a measure of the difference of $a \circ b$ and $a + b$:

$$a * b = 0 \iff a + b = a \circ b.$$

$*$ revisited

Let A be a skew brace, then $a * b := -a + a \circ b - b$ gives a measure of the difference of $a \circ b$ and $a + b$:

$$a * b = 0 \iff a + b = a \circ b.$$

Example

Let A^2 be the subgroup of $(A, +)$ generated by

$$\{a * b \mid a, b \in A\},$$

then A^2 is an ideal of A and A/A^2 is a trivial skew brace.

$*$ revisited

Let A be a skew brace, then $a * b := -a + a \circ b - b$ gives a measure of the difference of $a \circ b$ and $a + b$:

$$a * b = 0 \iff a + b = a \circ b.$$

Example

Let A^2 be the subgroup of $(A, +)$ generated by

$$\{a * b \mid a, b \in A\},$$

then A^2 is an ideal of A and A/A^2 is a trivial skew brace.

Note that A is trivial if and only if $A^2 = 0$.

Ideals and opposite skew braces

Proposition

The ideals of $A = (A, +, \circ)$ *and* $A_{\mathrm{op}} = (A, +_{\mathrm{op}}, \circ)$ *coincide.*

Example

We can apply the previous example to A_{op} in order to obtain the ideal A_{op}^2 of A_{op}, which is the subgroup of $(A, +_{\mathrm{op}})$ generated by

$$\{a *_{\mathrm{op}} b \mid a, b \in A\},$$

where $a *_{\mathrm{op}} b = -a +_{\mathrm{op}} a \circ b -_{\mathrm{op}} b = -b + a \circ b - a$. In this case A/A_{op}^2 is an almost trivial skew brace.

Ideals and opposite skew braces

Proposition

The ideals of $A = (A, +, \circ)$ and $A_{\mathrm{op}} = (A, +_{\mathrm{op}}, \circ)$ coincide.

Example

We can apply the previous example to A_{op} in order to obtain the ideal A^2_{op} of A_{op}, which is the subgroup of $(A, +_{\mathrm{op}})$ generated by

$$\{a *_{\mathrm{op}} b \mid a, b \in A\},$$

where $a *_{\mathrm{op}} b = -a +_{\mathrm{op}} a \circ b -_{\mathrm{op}} b = -b + a \circ b - a$. In this case A/A^2_{op} is an almost trivial skew brace.

Note that A is an almost trivial skew brace if and only if $A^2_{\mathrm{op}} = 0$.

Two-sided skew braces

Let A be a two-sided skew brace. Then for all $a, b, c, d \in A$,

$$\begin{aligned}(a+b)\circ(c+d) &= (a+b)\circ c-(a+b)+(a+b)\circ d\\ &= a\circ c-c+b\circ c-b-a+a\circ d-d+b\circ d\\ &= a\circ c+b *_{\mathrm{op}} c+a*d+b\circ d\end{aligned}$$

Two-sided skew braces

Let A be a two-sided skew brace. Then for all $a, b, c, d \in A$,

$$\begin{aligned}(a+b)\circ(c+d) &= (a+b)\circ c-(a+b)+(a+b)\circ d\\ &= a\circ c-c+b\circ c-b-a+a\circ d-d+b\circ d\\ &= a\circ c+b *_{\mathrm{op}} c+a*d+b\circ d\end{aligned}$$

but also

$$\begin{aligned}(a+b)\circ(c+d) &= a\circ(c+d)-(c+d)+b\circ(c+d)\\ &= a\circ c-a+a\circ d-d-c+b\circ c-b+b\circ d\\ &= a\circ c+a*d+b *_{\mathrm{op}} c+b\circ d.\end{aligned}$$

Two-sided skew braces

Hence $b *_{\mathrm{op}} c + a * d = a * d + b *_{\mathrm{op}} c$, therefore

Proposition ([T22])

Let A be a two-sided skew brace, then A^2 centralizes A^2_{op} in $(A, +)$.

Theorem ([T22])

Let A be a two-sided skew brace, then $(A^2 \cap A^2_{\mathrm{op}}, +)$ is abelian, so $A^2 \cap A^2_{\mathrm{op}}$ is a two-sided brace.

Natural question: what can we say about $A/(A^2 \cap A^2_{\mathrm{op}})$?

Weakly trivial skew braces

Definition

A skew brace A is called *weakly trivial* if $A^2 \cap A^2_{\mathrm{op}} = 0$.

Weakly trivial skew braces

Definition

A skew brace A is called *weakly trivial* if $A^2 \cap A^2_{\mathrm{op}} = 0$.

Lemma ([T22])

Every weakly trivial skew brace is two-sided.

Weakly trivial skew braces

Definition

A skew brace A is called *weakly trivial* if $A^2 \cap A^2_{\mathrm{op}} = 0$.

Lemma ([T22])

Every weakly trivial skew brace is two-sided.

Lemma ([T22])

Let A be a skew brace, then $A/(A^2 \cap A^2_{\mathrm{op}})$ is a weakly trivial skew brace.

Weakly trivial skew braces

Definition

A skew brace A is called *weakly trivial* if $A^2 \cap A^2_{\mathrm{op}} = \mathbf{0}$.

Lemma ([T22])

Every weakly trivial skew brace is two-sided.

Lemma ([T22])

Let A be a skew brace, then $A/(A^2 \cap A^2_{\mathrm{op}})$ is a weakly trivial skew brace.

Proposition ([T22])

Every two-sided skew brace is an extensions of a weakly trivial skew brace by a two-sided brace.

Weakly trivial skew braces

Example

- Trivial and almost trivial skew braces are weakly trivial.
- Direct products and sub skew braces of weakly trivial skew braces are weakly trivial.

Weakly trivial skew braces

Example

- Trivial and almost trivial skew braces are weakly trivial.
- Direct products and sub skew braces of weakly trivial skew braces are weakly trivial.

Proposition ([T22])

A weakly trivial skew brace embeds into a direct product of a trivial and almost trivial skew brace, more precisely

$$\iota : A \to A/A^2 \times A/A^2_{\mathrm{op}}, \quad a \mapsto (a + A^2, a + A^2_{\mathrm{op}}).$$

Properties of trivial and almost trivial skew braces

Let A be a trivial or almost trivial skew brace.

1. A quotient of a trivial, respectively almost trivial, skew brace is trivial, respectively almost trivial.

Properties of trivial and almost trivial skew braces

Let A be a trivial or almost trivial skew brace.

1. A quotient of a trivial, respectively almost trivial, skew brace is trivial, respectively almost trivial.
2. It is true that $(A,+) \cong (A,\circ)$.

Properties of trivial and almost trivial skew braces

Let A be a trivial or almost trivial skew brace.

1. A quotient of a trivial, respectively almost trivial, skew brace is trivial, respectively almost trivial.
2. It is true that $(A, +) \cong (A, \circ)$.
3. The nilpotency classes of $(A, +)$ and $(A, \circ)$ coincide. The same holds for solvability.

Properties of weakly trivial skew braces

Let A be a weakly trivial skew brace.

1. It is not true that any quotient of A is weakly trivial.

Properties of weakly trivial skew braces

Let A be a weakly trivial skew brace.

1. It is not true that any quotient of A is weakly trivial.
2. It is not true that $(A,+) \cong (A,\circ)$, the smallest counterexample has order 100.

Properties of weakly trivial skew braces

Let A be a weakly trivial skew brace.

1. It is not true that any quotient of A is weakly trivial.
2. It is not true that $(A,+) \cong (A,\circ)$, the smallest counterexample has order 100.
3. The nilpotency classes of $(A,+)$ and $(A,\circ)$ coincide. The same holds for solvability.

Consequences: simple two-sided skew braces

Definition

A skew brace A is *simple* if it contains no nontrivial ideals.

Theorem ([T22])

Let A be a simple two-sided skew brace, then one of the following holds:

1. $A \cong (G, \circ, \circ)$ *with* $(G, \circ)$ *a simple group,*
2. $A \cong (G, \circ_{\mathrm{op}}, \circ)$ *with* $(G, \circ)$ *a simple group,*
3. A *is an infinite two-sided brace.*

Consequences: connections $(A, +)$ and $(A, \circ)$

Theorem ([T22])

Let A be a two-sided skew brace and $(A, \circ)$ solvable of degree n, then $(A, +)$ is solvable of degree at most $n + 1$.

Consequences: connections $(A, +)$ and $(A, \circ)$

Theorem ([T22])

Let A be a two-sided skew brace and $(A, \circ)$ solvable of degree n, then $(A, +)$ is solvable of degree at most $n + 1$.

For general skew braces the implication

$$(A, \circ) \text{ solvable } \implies (A, +) \text{ solvable}$$

does not hold!

Consequences: connections $(A, +)$ and $(A, \circ)$

Conjecture (Byott)

Let A be a finite skew brace, if $(A, +)$ is solvable then also $(A, \circ)$ is solvable.

Consequences: connections $(A, +)$ and $(A, \circ)$

Conjecture (Byott)

Let A be a finite skew brace, if $(A, +)$ is solvable then also $(A, \circ)$ is solvable.

Theorem ([Nas19])

Byott's conjecture holds for two-sided skew braces.

Consequences: connections $(A, +)$ and $(A, \circ)$

Conjecture (Byott)

Let A be a finite skew brace, if $(A, +)$ is solvable then also $(A, \circ)$ is solvable.

Theorem ([Nas19])

Byott's conjecture holds for two-sided skew braces.

Is finiteness necessary?

Consequences: connections $(A, +)$ and $(A, \circ)$

Theorem ([T22, Wat68])

Let A be a two-sided skew brace, then $(A, +)$ satisfies the ACC on subgroups if and only if $(A, \circ)$ satisfies the ACC on subgroups.

Consequences: connections $(A,+)$ and $(A,\circ)$

Theorem ([T22, Wat68])

Let A be a two-sided skew brace, then $(A,+)$ satisfies the ACC on subgroups if and only if $(A,\circ)$ satisfies the ACC on subgroups.

Theorem ([T22])

Let A be a two-sided skew brace satisfying the equivalent conditions of the previous theorem, then $(A,+)$ is solvable if and only if $(A,\circ)$ is solvable.

L. Guarnieri and L. Vendramin.
Skew braces and the Yang–Baxter equation.
Mathematics of Computation, 86(307):2519–2534, 2017.

A. Koch and P. J. Truman.
Opposite skew left braces and applications.
Journal of Algebra, 546:218–235, 2020.

T. Nasybullov.
Connections between properties of the additive and the multiplicative groups of a two-sided skew brace.
Journal of Algebra, 540:156–167, 2019.

W. Rump.
Braces, radical rings, and the quantum Yang–Baxter equation.
Journal of Algebra, 307(1):153–170, 1 2007.

T.
On two-sided skew braces.
In preparation, 2022.

J. F. Watters.
On the adjoint group of a radical ring.
Journal of the London Mathematical Society, s1-43(1):725–729, 1968.