Dimension-free estimates in analysis: Hardy-Littlewood maximal functions

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Many of the linear and sublinear operators of mathematical analysis are defined on functions on $\mathbb{R}^d$ where d denotes the dimension. This pertains for example to Riesz transforms and Hardy-Littlewood maximal functions. At first sight it often seems that the norms of these operators grow exponentially with the dimension d. The rapid growth of these norms is similar to a phenomenon labelled by R. E. Bellman the 'curse of dimensionality', which appears often in numerical analysis, combinatorics, machine learning, data mining and databases.

However, for Riesz transforms and Hardy-Littlewood maximal functions, it has been proved that in many cases the growth of the dimension d does not necessarily force the growth of norms of these operators. This phenomenon has been labelled 'dimension-free estimates'. Studies of dimension-free estimates in harmonic analysis were initiated by E. M. Stein in 1980'. Research in this topic was conducted by a number of prominent mathematicians over the years. Still, a number of major open problems in the area remain unsolved, some of them for more than 30 years.

The Hardy-Littlewood maximal function is one of the most important non-linear operators in harmonic analysis and real analysis. The aim of the lecture is to discuss various problems related with dimension-free bounds for Hardy-Littlewood maximal functions, both in the continuous and discrete settings. In the continuous setting these problems are connected with high-dimensional convex geometry and local theory of Banach spaces. In the discrete setting, with $\mathbb{R}^d$ being replaced by $\mathbb{Z}^d$, connections with number theory and combinatorics play an important role. The studies of dimension-free estimates for Hardy-Littlewood maximal operators in the discrete setting were initiated recently in my collaboration with J. Bourgain, M. Mirek, and E.M. Stein.

The lecture will be divided into two parts. The purpose of the first part is to gently introduce the audience to the topic and to present the state of the art both in the continuous and in the discrete settings. In the second part we shall highlight some of the methods and difficulties in the discrete case. We will achieve this by comparing the discrete maximal function defined over the cubes with the one defined over the Euclidean balls.