What is K-theory and what is it good for?

Baby Steps Meeting

September 9, 2021

- The basic definition of K-theory
- A brief history of K-theory
- Algebraic versus topological K-theory
- The unity of K-theory

Dedicated to the memory of Sir Michael Atiyah

Let J be an abelian semi-group.

$\widehat{J}$ denotes the abelian group :

$$\widehat{J} \quad = \quad J \bigoplus J/\sim$$

$$(\xi,\eta) \sim (\xi',\eta') \quad \Longleftrightarrow \quad \exists \quad \theta \in J \quad \text{with}$$

$$\xi + \eta' + \theta = \xi' + \eta + \theta$$

<u>Example</u> $\mathbb{N} = \{1,2,3,\ldots\}$

$$\widehat{\mathbb{N}} = \mathbb{Z}$$

Let Λ be a ring with unit 1_Λ.

$M_n(\Lambda)$ denotes the ring of all $n \times n$ matrices

$[a_{ij}]$ with each $a_{ij} \in \Lambda$. $\qquad n = 1, 2, 3, \ldots$

$M_n(\Lambda)$ is again a ring with unit.

$GL(n, \Lambda) = \{$ invertible elements of $M_n(\Lambda)\}$

$P_n(\Lambda) = \{\alpha \in M_n(\Lambda) | \alpha^2 = \alpha\} \quad n = 1, 2, 3 \ldots$

Definition

$\alpha, \beta \in P_n(\Lambda)$ are similar if $\exists \gamma \in GL(n, \Lambda)$ with $\gamma\alpha\gamma^{-1} = \beta$.

Set $P(\Lambda) = P_1(\Lambda) \cup P_2(\Lambda) \cup P_3(\Lambda) \cup \ldots$

Impose an equivalence relation stable similarity on $P(\Lambda)$.

Notation. If r is a non-negative integer $[0_r]$ is the $r \times r$ zero matrix.

Definition

$\alpha \in P_n(\Lambda)$ and $\beta \in P_m(\Lambda)$ are stably similar iff there exist non-negative integers r, s with $n + r = m + s$ and with

$$\alpha \oplus [0_r] \text{ is similar to } \beta \oplus [0_s]$$

$J(\Lambda) = P(\Lambda)/$(stable similarity)

$J(\Lambda)$ is an abelian semi-group.

$\alpha + \beta =$

α	0
0	β

Definition

$\alpha \in P_n(\Lambda)$ and $\beta \in P_m(\Lambda)$ are stably similar iff there exist non-negative integers r, s with $n + r = m + s$ and with

$$\begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{r} \\ \end{array}$$

$$\left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & 0 \end{array}\right] \text{ and } \left[\begin{array}{c|c} \beta & 0 \\ \hline 0 & 0 \end{array}\right] \text{ similar}$$

(with column blocks of widths n, r for the first matrix and m, s for the second)

Set $J(\Lambda) = P(\Lambda)/(\text{stable similarity})$.

$J(\Lambda) = P(\Lambda)/$(stable similarity)

$J(\Lambda)$ is an abelian semi-group.

$\alpha + \beta =$

α	0
0	β

Definition

$K_0\Lambda = \widehat{J(\Lambda)}$

This is the basic definition of K-theory.

Λ, Ω rings with unit

$\varphi\colon \Lambda \to \Omega$ ring homomorphism with $\varphi(1_\Lambda) = 1_\Omega$

$\varphi_*\colon K_0\Lambda \to K_0\Omega$

$\varphi_*[a_{ij}] = [\varphi(a_{ij})]$

$\varphi\colon K_0\Lambda \to K_0\Omega$ is a homomorphism of abelian groups

Example

If Λ is a field, then $[a_{ij}], [b_{kl}]$ in $P(\Lambda)$ are stably similar iff

$$\text{rank}[a_{ij}] = \text{rank}[b_{kl}],$$

where the rank of an $n \times n$ matrix is the dimension (as a vector space over Λ) of the sub vector space of $\Lambda^n = \Lambda \oplus \cdots \oplus \Lambda$ spanned by the rows of the matrix.

Hence if Λ is a field, $J(\Lambda) = \{0, 1, 2, 3, \ldots\}$ and $K_0\Lambda = \mathbb{Z}$.

X compact Hausdorff topological space

$C(X) = \{\alpha\colon X \to \mathbb{C} | \alpha \text{ is continuous}\}$

$C(X)$ is a ring with unit.

$(\alpha + \beta)x = \alpha(x) + \beta(x)$

$(\alpha\beta)x = \alpha(x)\beta(x) \qquad x \in X, \quad \alpha, \beta \in C(X)$

The unit is the constant function 1.

Definition (M. Atiyah - F. Hirzebruch)

Let X be a compact Hausdorff topological space.

$$K^0(X) = K_0 C(X)$$

Example

$$S^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1^2 + t_2^2 + t_3^2 = 1\}$$

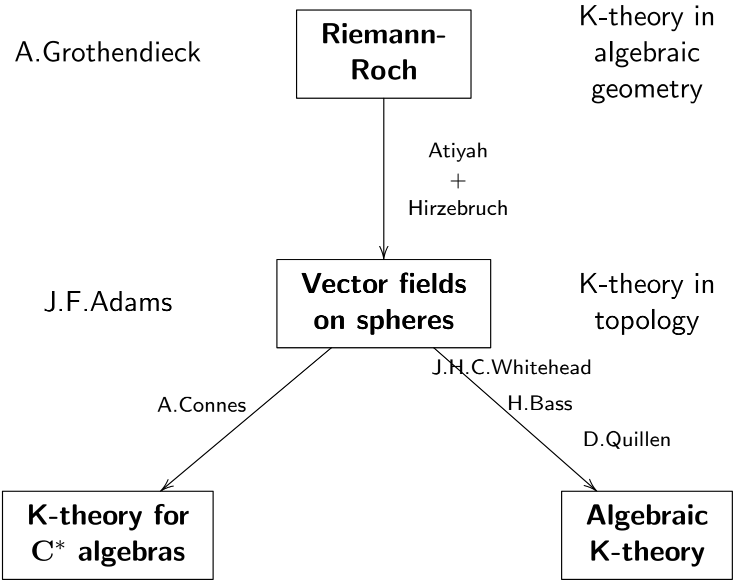
$$x_j \in C(S^2) \qquad x_j(t_1, t_2, t_3) = t_j \qquad j = 1, 2, 3$$

$$K_0 C(S^2) = \mathbb{Z} \oplus \mathbb{Z}$$

$$[1] \qquad \begin{bmatrix} \frac{1+x_3}{2} & \frac{x_1+ix_2}{2} \\ \frac{x_1-ix_2}{2} & \frac{1-x_3}{2} \end{bmatrix}$$

$i = \sqrt{-1}$

A brief history of K-theory



HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E}$ = sheaf of germs of algebraic sections of E

$H^j(M, \underline{E})$:= j-th cohomology of M using $\underline{E}$, $j = 0, 1, 2, 3, \ldots$

Lemma

For all $j = 0, 1, 2, \ldots \dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$.

For all $j > \dim_{\mathbb{C}}(M), \quad H^j(M, \underline{E}) = 0$.

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

Theorem (HRR)

Let M be a non-singular projective algebraic variety / $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$


1, ε, ε², ε³, ε⁴
5 Eckp. des reg 5-Ecks
A5
{ (a b / c d) | ad - bc = 1, a, b, c, d ∈ F5 }.

Tangent Vector Fields on Spheres

S^{n-1} denotes the unit sphere of $\mathbb{R}^n$

$S^{n-1} = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \mid t_1^2 + t_2^2 + \cdots + t_n^2 = 1\}$

A continuous tangent vector field V on S^{n-1} can be viewed as a continuous function

$$V : S^{n-1} \to \mathbb{R}^n$$

such that every $p \in S^{n-1}$ has $V(p)$ perpendicular to p.

$$V(p) \perp p$$

Definition

Continuous tangent vector fields $V_1, V_2, \ldots, V_r$ on S^{n-1} are linearly independent if for every $p \in S^{n-1}$, $V_1(p), V_2(p), \ldots, V_r(p)$ are linearly independent in $\mathbb{R}^n$.

$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
usual inner product
$\langle (t_1, t_2, \ldots, t_n), (s_1, s_2, \ldots, s_n) \rangle = t_1 s_1 + t_2 s_2 + \cdots + t_n s_n$
$p \in \mathbb{R}^n, q \in \mathbb{R}^n$ $p \perp q \Leftrightarrow \langle p, q \rangle = 0.$

$S^{n-1} \subset \mathbb{R}^n$
A continuous tangent vector field V on S^{n-1} an be viewed as a continuous function

$$V \colon S^{n-1} \to \mathbb{R}^n$$

such that $\forall p \in S^{n-1}$

$$p \perp V(p).$$

Problem

What is the maximum r such that S^{n-1} admits r linearly independent continuous tangent vector fields?

Example

For $S^2, S^4, S^6, S^8, \ldots, r = 0$.
If V is a continuous tangent vector field on an even-dimensional sphere, then for at least one point p in the sphere $V(p) = 0$.

Theorem (J. F. Adams)

Set

$$n = 2^{c(n)} 16^{d(n)} u, \quad 0 \leq c(n) \leq 3, \quad u \text{ odd.}$$

Define $\rho(n)$ by $\rho(n) = 2^{c(n)} + 8d(n)$.

Then S^{n-1} admits $\rho(n) - 1$ linearly independent continuous tangent vector fields and S^{n-1} does not admit $\rho(n)$ linearly independent continuous tangent vector fields.

Reference. J. F. Adams "Vector fields on spheres" Ann. of Math. **75** (1962), 603-632

Topic 2: C^* algebras

Definition

A Banach algebra is an algebra A over $\mathbb{C}$ with a given norm $\| \; \|$

$$\| \; \| : A \to \{t \in \mathbb{R} \mid t \geqq 0\}$$

such that A is a complete normed algebra:

$\|\lambda a\| = |\lambda|\|a\| \quad \lambda \in \mathbb{C}, \quad a \in A$

$\|a + b\| \leq \|a\| + \|b\| \quad a, b \in A$

$\|ab\| \leq \|a\|\|b\| \qquad a, b \in A$

$\|a\| = 0 \iff a = 0$

Every Cauchy sequence is convergent in A (with respect to the metric $\|a - b\|$).

C^* algebras

$A \quad C^*$ algebra $\qquad *\colon A \to A$

$A = (A, \|\ \|, *) \qquad a \mapsto a^*$

$(A, \|\ \|)$ is a Banach algebra

$(a^*)^* = a$

$(a+b)^* = a^* + b^*$

$(ab)^* = b^*a^*$

$(\lambda a)^* = \overline{\lambda} a^* \quad a, b \in A, \quad \lambda \in \mathbb{C}$

$\|aa^*\| = \|a\|^2 = \|a^*\|^2$

A $*$-homomorphism is an algebra homomorphism $\varphi\colon A \to B$ such that $\varphi(a^*) = (\varphi(a))^*\ \forall a \in A$.

Lemma

If $\varphi\colon A \to B$ is a $*$-homomorphism, then $\|\varphi(a)\| \leqq \|a\|\ \forall a \in A$.

Example

X topological space, Hausdorff, locally compact

$$\begin{aligned} X^+ &= \text{ one-point compactification of } X \\ &= X \cup \{p_\infty\} \\ C_0(X) &= \big\{\alpha\colon X^+ \to \mathbb{C} \mid \alpha \text{ continuous}, \alpha(p_\infty) = 0\big\} \\ \|\alpha\| &= \sup_{p \in X} |\alpha(p)| \\ \alpha^*(p) &= \overline{\alpha(p)} \\ (\alpha + \beta)(p) &= \alpha(p) + \beta(p) \quad p \in X \\ (\alpha\beta)(p) &= \alpha(p)\beta(p) \\ (\lambda\alpha(p) &= \lambda\alpha(p) \quad \lambda \in \mathbb{C} \end{aligned}$$

If X is compact Hausdorff, then

$$C_0(X) = C(X) = \{\alpha\colon X \to \mathbb{C} \mid \alpha \text{ continuous}\}$$

Example

H separable Hilbert space
separable = H admits a countable (or finite) orthonormal basis.

$$\mathcal{L}(H) = \{\text{bounded operators } T \colon H \to H\}$$

$$\|T\| = \sup_{\substack{u \in H \\ \|u\|=1}} \|Tu\| \qquad \underset{\|u\|=\langle u,u\rangle^{1/2}}{\text{operator norm}}$$

$$T^* = \text{ adjoint of } T \qquad \underset{u,v \in H}{\langle Tu, v\rangle = \langle u, T^* v\rangle}$$

$$(T+S)u = Tu + Su$$

$$(TS)u = T(Su)$$

$$(\lambda T)u = \lambda(Tu) \qquad \lambda \in \mathbb{C}$$

G topological group

locally compact

Hausdorff

second countable

(second countable = The topology of G has a countable base.)

Examples

Lie groups ($\pi_0(G)$ finite)	$\mathsf{SL}(n,\mathbb{R})$
p-adic groups	$\mathsf{SL}(n,\mathbb{Q}_p)$
adelic groups	$\mathsf{SL}(n,\mathbb{A})$
discrete groups	$\mathsf{SL}(n,\mathbb{Z})$

G topological group

locally compact

Hausdorff

second countable

Example

C_r^*G the reduced C^* algebra of G
Fix a left-invariant Haar measure dg for G
"left-invariant" $=$ whenever $f\colon G \to \mathbb{C}$ is continuous and compactly supported

$$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \forall \gamma \in G$$

L^2G Hilbert space
$L^2G = \{u\colon G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty\}$
$\langle u, v\rangle = \int_G \overline{u(g)}v(g)dg \qquad u, v \in L^2G$

$\mathcal{L}(L^2G) = C^*$ algebra of all bounded operators $T\colon L^2G \to L^2G$

$C_cG = \{f\colon G \to \mathbb{C} \mid f$ is continuous and f has compact support$\}$

C_cG is an algebra

$(\lambda f)g = \lambda(fg) \qquad \lambda \in \mathbb{C} \quad g \in G$

$(f+h)g = fg + hg$

Multiplication in C_cG is convolution

$$(f * h)g_0 = \int_G f(g)h(g^{-1}g_0)dg \qquad g_0 \in G$$

$0 \to C_c G \to \mathcal{L}(L^2 G)$

Injection of algebras

$f \mapsto T_f$

$T_f(u) = f * u \qquad u \in L^2 G$

$(f * u)g_0 = \int_G f(g)u(g^{-1}g_0)dg \qquad g_0 \in G$

$C_r^* G \subset \mathcal{L}(L^2 G)$

$C_r^* G = \overline{C_c G} =$ closure of $C_c G$ in the operator norm

$C_r^* G$ is a sub C^* algebra of $\mathcal{L}(L^2 G)$

A C^* algebra (or a Banach algebra) with unit 1_A.

Define abelian groups $K_1A, K_2A, K_3A, ...$ as follows :

$\mathrm{GL}(n, A)$ is a topological group.

The norm $\| \|$ of A topologizes $\mathrm{GL}(n, A)$.

$\mathrm{GL}(n, A)$ embeds into $\mathrm{GL}(n+1, A)$.

$$\mathrm{GL}(n, A) \hookrightarrow \mathrm{GL}(n+1, A)$$

$$\begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & ... & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & ... & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & ... & a_{nn} & 0 \\ 0 & ... & 0 & 1_A \end{bmatrix}$$

$\mathrm{GL}\, A = \lim\limits_{n\to\infty} \mathrm{GL}(n, A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A)$

$\mathrm{GL}\,A = \lim_{n\to\infty} \mathrm{GL}(n, A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A)$

Give $\mathrm{GL}\,A$ the direct limit topology.

This is the topology in which a set $U \subset \mathrm{GL}\,A$ is open if and only if $U \cap \mathrm{GL}(n, A)$ is open in $\mathrm{GL}(n, A)$ for all $n = 1, 2, 3, \ldots$

A C^* algebra (or a Banach algebra) with unit 1_A

$K_1 A, K_2 A, K_3 A, \ldots$

Definition

$K_j A := \pi_{j-1}(\mathrm{GL}\, A)$ $\qquad$ $j = 1, 2, 3, \ldots$

$\Omega^2\, \mathrm{GL}\, A \sim \mathrm{GL}\, A$ $\qquad$ Bott Periodicity

$K_j A \cong K_{j+2} A$ $\qquad$ $j = 0, 1, 2, \ldots$

$K_0 A \qquad K_1 A$

A C^* algebra (or a Banach algebra) with unit 1_A

$K_0A = K_0^{alg}A = \widehat{J(A)}$

$A = (A, \| \|, *)$

For K_0A forget $\| \|$ and $*$. View A as a ring with unit.

Define K_0A as above using idempotent matrices.

For K_1A cannot forget $\| \|$ and $*$.

K_0A $\qquad\qquad$ K_1A

A C^* algebra (or a Banach algebra) with unit 1_A

The Bott periodicity isomorphism

$K_0 A = \widehat{J(A)} \longrightarrow K_2 A = \pi_1 GLA$

assigns to $\alpha \in P_n(A)$ the loop of $n \times n$ invertible matrices

$t \mapsto I + (e^{2\pi i t} - 1)\alpha \qquad\qquad t \in [0,1]$

$I =$ the $n \times n$ identity matrix

$A \quad C^*$ algebra (or a Banach algebra)

If A is not unital, adjoin a unit.

$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{C} \longrightarrow 0$

Define: $K_j A = K_j \tilde{A}$ $\quad j = 1, 3, 5, \ldots$

$K_j A =$ Kernel$(K_j \tilde{A} \longrightarrow K_j \mathbb{C})$ $\quad j = 0, 2, 4, \ldots$

$K_j A \cong K_{j+2} A$ $\quad j = 0, 1, 2, \ldots$

$K_0 A \qquad K_1 A$

FUNCTORIALITY OF K-THEORY

A, B C^* algebras

$\varphi : A \longrightarrow B$ $*$- homomorphism

$\varphi_* : K_j A \longrightarrow K_j B$ $j = 0, 1$

K-theory can be applied to classification problems for C^* algebras. See results of G. Elliott and others.

SIX TERM EXACT SEQUENCE

Let

$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$

be a short exact sequence of C^* algebras.

Then there is a six-term exact sequence of abelian groups

$$\begin{array}{ccccc} K_0 I & \longrightarrow & K_0 A & \longrightarrow & K_0 B \\ \uparrow & & & & \downarrow \\ K_1 B & \longleftarrow & K_1 A & \longleftarrow & K_1 I \end{array}$$

G topological group

locally compact

Hausdorff

second countable

(second countable = topology of G has a countable base)

C_r^*G the reduced C^* algebra of G

Problem

$K_j C_r^* G = ? \quad j = 0, 1$

Conjecture (P. Baum - A. Connes)

$$\mu\colon K_j^G(\underline{E}G) \to K_j C_r^* G \text{ is an isomorphism.} \quad j = 0, 1$$

G compact or G abelian

$$\Downarrow$$

Conjecture true

Corollaries of BC

Novikov conjecture

Stable Gromov Lawson Rosenberg conjecture (Hanke + Schick)

Idempotent conjecture

Kadison Kaplansky conjecture

Mackey analogy (Higson)

Construction of the discrete series via Dirac induction
(Parthasarathy, Atiyah + Schmid, V. Lafforgue)

Homotopy invariance of ρ-invariants
(Keswani, Piazza + Schick)

G topological group

locally compact

Hausdorff

second countable

(second countable = The topology of G has a countable base.)

Examples

Lie groups ($\pi_0(G)$ finite)	$\mathrm{SL}(n, \mathbb{R})$ OK✓
p-adic groups	$\mathrm{SL}(n, \mathbb{Q}_p)$OK✓
adelic groups	$\mathrm{SL}(n, \mathbb{A})$OK✓
discrete groups	$\mathrm{SL}(n, \mathbb{Z})$

Theorem (N. Higson + G. Kasparov)

Let Γ be a discrete (countable) group which is amenable or a-t-menable. Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma) \to K_j C_r^* \Gamma$$

is an isomorphism. $j = 0, 1$

Theorem (G. Yu + I. Mineyev, V. Lafforgue)

Let Γ be a discrete (countable) group which is hyperbolic (in Gromov's sense). Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma) \to K_j C_r^* \Gamma$$

is an isomorphism. $j = 0, 1$

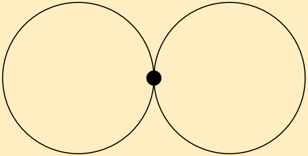
$\mathrm{SL}(3,\mathbb{Z})$??????

Γ group (not topologized = discrete group)

$$B\Gamma \quad \begin{cases} \text{triangulable topological space} \\ \text{connected} \\ \pi_1(B\Gamma) = \Gamma \\ \pi_j(B\Gamma) = 0 \quad \text{for all } j > 1 \end{cases}$$

Example

F_2 = Free group on two generators

$BF_2 =$ 

Example

$\mathbb{Z}/2\mathbb{Z}$

$$B(\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}P^\infty = \lim_{n\to\infty} \mathbb{R}P^n$$

Algebraic K Theory
Λ ring with unit 1_Λ
$K_0\Lambda$
$\mathrm{GL}\,\Lambda = \lim_{n\to\infty} \mathrm{GL}(n,\Lambda)$

Definition

$K_1^{alg}\Lambda = \mathrm{GL}\,\Lambda/[\mathrm{GL}\,\Lambda, \mathrm{GL}\,\Lambda]$

Lemma

$[\mathrm{GL}\,\Lambda, \mathrm{GL}\,\Lambda]$ is perfect.

$B\,\mathrm{GL}\,\Lambda$
$(B\,\mathrm{GL}\,\Lambda)_+$ $\quad + $ -construction D. Quillen

$$B\,\mathrm{GL}\,\Lambda \longrightarrow (B\,\mathrm{GL}\,\Lambda)_+$$

$$\pi_1(B\,\mathrm{GL}\,\Lambda)_+ = \mathrm{GL}\,\Lambda/[\mathrm{GL}\,\Lambda, \mathrm{GL}\,\Lambda]$$
$$H_j B\,\mathrm{GL}\,\Lambda = H_j(B\,\mathrm{GL}\,\Lambda)_+ \quad j = 0, 1, 2, 3, \ldots$$

Definition

$$K_j^{alg}\Lambda = \pi_j(B\,\mathrm{GL}\,\Lambda)_+ \qquad j = 1, 2, 3, \ldots$$

Extend to the case when Λ does not have a unit

$$0 \to \Lambda \to \widetilde{\Lambda} \to \mathbb{Z} \to 0$$

$$K_j^{alg}\Lambda = \text{Kernel}\left(K_j^{alg}\widetilde{\Lambda} \to K_j^{alg}\mathbb{Z}\right)$$

Lichtenbaum Conjecture: Special Case

Conjecture

Let F be a totally real algebraic number field. Then for $n = 2, 4, 6, \ldots$

$$\zeta_F(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_F)|}{|K_{2n-1}(\mathcal{O}_F)|}$$

up to powers of 2.

Let F be an algebraic number field (F is a finite extension of $\mathbb{Q}$)

$\mathcal{O}_F$ denotes the ring of integers in F

Lichtenbaum Conjecture For $n \geqq 2$

$$\zeta_F(1-n) = \pm \frac{|K_{2n-2}(\mathcal{O}_F)|}{|K_{2n-1}(\mathcal{O}_F)_{tors}|} \cdot R_n^B(F)$$

up to powers of 2.

K **theory for C^* algebras**
A C^* algebra
trivial move = stabilizing A

$$M_n(A) \hookrightarrow M_{n+1}(A)$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

This is a one-to-one $*$-homomorphism $\therefore$ This is norm preserving

$$\begin{aligned} M_\infty(A) &= \lim_{\rightarrow} M_n(A) \\ &= \left\{ \begin{matrix} a_{11} a_{12} & \dots \\ a_{21} a_{22} & \dots \\ \vdots \ \vdots & \end{matrix} \ \middle| \ \text{Almost all } a_{ij} = 0 \right\} \end{aligned}$$

$\dot{A} = \overline{M_\infty(A)}$
$\dot{A}$ is the stabilization of A
$K_j(\dot{A}) = K_j(A) \qquad j = 0, 1$

Karoubi Conjecture

Let A be a C^* algebra, then

$$K_j(\dot{A}) = K_j^{alg}(\dot{A})$$

C^*-algebra K theory	Algebraic K-theory

The Karoubi conjecture was proved by A. Suslin and M. Wodzicki

Theorem (A. Suslin and M. Wodzicki)

Let A be a C^* algebra. Then

$$K_j\dot{A} = K_j^{alg}\dot{A} \qquad j = 0, 1, 2, 3, \ldots$$

$\dot{A}$ is the stabilization of A.

This theorem is the unity of K-theory. It says that C^* algebra K-theory is a pleasant subdiscipline of algebraic K-theory in which Bott periodicity is valid and certain basic examples are easy to calculate.

Example

Let H be a separable (but not finite dimensional) Hilbert space. i.e. H has a countable (but not finite) orthonormal basis

$$\dot{\mathbb{C}} = \mathcal{K} \subset \mathcal{L}(H)$$

$\mathcal{K} =$ The compact operators on H

$$K_j\mathbb{C} = K_j\dot{\mathbb{C}}$$

C^* algebra K theory

$$K_j^{alg}(\dot{\mathbb{C}}) = \begin{Bmatrix} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{Bmatrix}$$

algebraic K theory

$$\begin{pmatrix} \text{Commutative} \\ C^* \text{ algebras} \end{pmatrix} \sim \begin{pmatrix} \text{Locally compact Hausdorff} \\ \text{topological spaces} \end{pmatrix}^{op}$$

$$C_0(X) \leftarrow\!\shortmid X$$

$$C_0(X) := \{\alpha\colon X^+ \to \mathbb{C} \mid \alpha \text{ is continuous and } \alpha(p_\infty) = 0\}$$

$$X^+ = X \cup \{p_\infty\} \text{ is the one point compactification of } X$$

$$\begin{aligned} &f\colon X^+ \to Y^+ \\ &f(p_\infty) = q_\infty \\ &C_0(X) \leftarrow C_0(Y) \\ &\alpha \circ f \leftarrow\!\shortmid \alpha \in C_0(Y) \end{aligned}$$